SHARP GLOBAL WELL-POSEDNESS FOR A HIGHER ORDER SCHRÖDINGER EQUATION

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Abstract. Using the theory of almost conserved energies and the "I-method" developed by Colliander, Keel, Staffilani, Takaoka and Tao, we prove that the initial value problem for a higher order Schrödinger equation is globally well-posed in Sobolev spaces of order $s > 1/4$. This result is sharp.

1. Introduction

In this paper we will describe a sharp result of global well-posedness for solutions of the initial value problem (IVP)

$$\begin{cases} \partial_t u + ia\,\partial_x^2 u + b\,\partial_x^3 u + ic\,|u|^2 u + d\,|u|^2\partial_x u + e\,u^2\partial_x \bar{u} = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = \varphi(x), \end{cases} \tag{1.1}$$

where u is a complex valued function and a, b, c, d and e are real parameters with $be \neq 0$.

This model was proposed by Hasegawa and Kodama in [17, 21] to describe the nonlinear propagation of pulses in optical fibers. In literature, this model is called as a higher order nonlinear Schrödinger equation or also Airy-Schrödinger equation.

We consider the following gauge transformation

$$v(x,t) = \exp\Big(i\lambda\, x + i(a\,\lambda^2 - 2b\,\lambda^3)t\Big)\, u(x + (2a\,\lambda - 3b\lambda^2)t, t), \tag{1.2}$$

then, u solves (1.1) if and only if v satisfies the IVP

$$\begin{cases} \partial_t v + i(a - 3\lambda\, b)\partial_x^2 v + b\,\partial_x^3 v + i(c - \lambda(d-e))\,|v|^2 v + d\,|v|^2\partial_x v + e\,v^2\partial_x \bar{v} = 0, \\ v(x,0) = \exp(i\lambda\, x)\, u(x,0). \end{cases} \tag{1.3}$$

Thus, if we take $\lambda = a/3b$ in (1.2) and $c = (d-e)a/3b$, then the function

$$v(x,t) = \exp\Big(i\frac{a}{3b}\, x + i\frac{a^3}{27b^2}\, t\Big)\, u(x + \frac{a^2}{3b}\, t, t), \tag{1.4}$$

satisfies the complex modified Korteweg-de Vries type equation

$$\begin{cases} \partial_t v + b\,\partial_x^3 v + d\,|v|^2\partial_x v + e\,v^2\partial_x \bar{v} = 0, \\ v(x,0) = \exp(ia\,x/3b)\, u(x,0). \end{cases} \tag{1.5}$$

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Was shown in [22] that the flow associated to the IVP (1.1) leaves the following quantity

$$I_1(u)=\int_{\mathbb{R}}|u|^2(x,t)\,dx, \tag{1.6}$$

conserved in time. Also, when $be\neq 0$ we have the following conserved quantity

$$I_2(u)=c_1\int_{\mathbb{R}}|\partial_x u|^2(x,t)\,dx+c_2\int_{\mathbb{R}}|u|^4(x,t)dx+c_3\mathrm{Im}\int_{\mathbb{R}}u(x,t)\partial_x\overline{u(x,t)}dx, \tag{1.7}$$

where $c_1=3be$, $c_2=-e(e+d)/2$ and $c_3=(3bc-a(e+d))$. We may suppose $c_3=0$. In fact, when $c_3\neq 0$ we can take in the gauge transformation (1.2)

$$\lambda=-\frac{c_3}{6be}.$$

Then, u solves (1.1) if and only if v satisfies (1.3) and in this new IVP we have the constant $c_3=0$.

We say that the IVP (1.1) is locally well-posed in X (Banach space) if the solution uniquely exists in certain time interval $[-T,T]$ (unique existence), the solution describes a continuous curve in X in the interval $[-T,T]$ whenever initial data belongs to X (persistence), and the solution varies continuously depending upon the initial data (continuous dependence) i.e. continuity of application $u_0\mapsto u(t)$ from X to $\mathcal{C}([-T,T];X)$. We say that the IVP (1.1) is globally well-posed in X if the same properties hold for all time $T>0$. If some hypotheses in the definition of local well-posed fail, we say that the IVP is ill-posed.

Particular cases of (1.1) are the following:

• Cubic nonlinear Schrödinger equation (NLS), ($a=\mp 1$, $b=0$, $c=-1$, $d=e=0$).

$$iu_t\pm u_{xx}+|u|^2u=0,\quad x,t\in\mathbb{R}. \tag{1.8}$$

The best known local result for the IVP associated to (1.8) is in $H^s(\mathbb{R})$, $s\geq 0$, obtained by Tsutsumi [31]. Since the L^2 norm is preserved in (1.8), one has that (1.8) is globally well-posed in $H^s(\mathbb{R})$, $s\geq 0$.

• Nonlinear Schrödinger equation with derivative ($a=-1$, $b=0$, $c=0$, $d=2e$).

$$iu_t+u_{xx}+i\lambda(|u|^2u)_x=0,\quad x,t\in\mathbb{R}. \tag{1.9}$$

The best known local result for the IVP associated to (1.9) is in $H^s(\mathbb{R})$, $s\geq 1/2$, obtained by Takaoka [30]. Colliander et al. [10] they proved that (1.9) is globally well-posed in $H^s(\mathbb{R})$, $s>1/2$.

• Complex modified Korteweg-de Vries (mKdV) equation ($a=0$, $b=1$, $c=0$, $d=1$, $e=0$).

$$u_t+u_{xxx}+|u|^2u_x=0,\quad x,t\in\mathbb{R}. \tag{1.10}$$

If u is real, (1.10) is the usual mKdV equation. Kenig et al. [19] proved that the IVP associated to it is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1/4$ and Colliander et al. [11], proved that (1.10) is globally well-posed in $H^s(\mathbb{R})$, $s > 1/4$.

• When $a \neq 0$ is real and $b = 0$, we obtain a particular case of the well-known mixed nonlinear Schrödinger equation

$$u_t = iau_{xx} + \lambda(|u|^2)_x u + g(u), \quad x, t \in \mathbb{R}, \tag{1.11}$$

where g satisfies some appropriate conditions and $\lambda \in \mathbb{R}$ is a constant. Ozawa and Tsutsumi in [24] proved that for any $\rho > 0$, there is a positive constant $T(\rho)$ depending only on ρ and g, such that the IVP (1.11) is locally well-posed in $H^{1/2}(\mathbb{R})$, whenever the initial data satisfies

$$\|u_0\|_{\mathrm{H}^{1/2}} \leq \rho.$$

There are other dispersive models similar to (1.1), see for instance [1, 8, 25, 26, 28] and the references therein.

Regarding the IVP (1.1), Laurey in [22] showed that the IVP is locally well-posed in $H^s(\mathbb{R})$ with $s > 3/4$, and using the quantities (1.6) and (1.7) she proved the global well-posedness in $H^s(\mathbb{R})$ with $s \geq 1$. In [27] Staffilani established the local well-posedness in $H^s(\mathbb{R})$ with $s \geq 1/4$, for the IVP associated to (1.1), improving Laurey's result.

In the IVP (1.1), when a, b are real functions of t, in [4, 6] was prove the local well-posedness in $H^s(\mathbb{R})$, $s \geq 1/4$. Also, in [4, 7] was study the unique continuation property for the solution of (1.1).

Remark 1.1. 1) *Using* (1.4) *and the results obtained in* [11] *we have that the PVI* (1.1) *is globally well-posed in $H^s(\mathbb{R})$ with $s > 1/4$, for initial data of the form:*

$$\exp\{-i\frac{a}{3b}x\}v_0(x), \quad \exp\{-i\frac{a}{3b}x\}(v_0(x) + iv_0(x)),$$

where $v_0 \in H^s$, $s > 1/4$, $v_0 \in \mathbb{R}$. Therefore it suggests us to improve the result and obtain the global existence for the general case in $H^s(\mathbb{R})$, $s > 1/4$.

2) *If $e = 0$, $bd > 0$ and $c = (a/3b)d$ in (1.1), then the equation*

$$\partial_t u + ia\partial_x^2 u + b\partial_x^3 u + i\frac{a}{3b}d|u|^2 u + d|u|^2\partial_x u = 0, \tag{1.12}$$

have the following solution with two parameters

$$u_{\eta,N}(x,t) = f_\eta(x + \psi(\eta, N)t)\exp i\{Nx + \phi(\eta, N)t\}, \tag{1.13}$$

where $f_\eta(x) = \eta f(\eta x)$, $f(x) = (A\cosh x)^{-1}$, $A = \sqrt{d/(6b)}$, $\psi(\eta, N) = 2aN + 3bN^2 - \eta^2 b$ and $\phi(\eta, N) = aN^2 + bN^3 - 3\eta^2 bN - a\eta^2$.

Using the transformation (1.4) we can to obtain other family of solutions for (1.12). In fact, let w *solution of*

$$\begin{cases} \partial_t w + \partial_x^3 w + |w|^2 \partial_x w = 0, \quad x, t \in \mathbb{R}, \\ w(x,0) = w_0(x) = f_1(x) \exp i\{Nx\} = (\frac{1}{\sqrt{6}} \cosh x)^{-1} \exp i\{Nx\}, \end{cases} \tag{1.14}$$

given by (1.13). If w *is a solution of (1.14), then*

$$v(x,t) = \frac{1}{\alpha} w(b^{-1/3}x, t), \quad \alpha = \sqrt{\frac{d}{b^{1/3}}}$$

is a solution of

$$\begin{cases} \partial_t v + b\partial_x^3 v + d|v|^2 \partial_x v = 0, \quad x, t \in \mathbb{R}, \\ v(x,0) = v_0(x), \end{cases} \tag{1.15}$$

with initial data $v_0(x) = (1/\alpha) w(b^{-1/3}x, 0)$ *and if* v *is a solution of (1.15) then, using the transformation (1.4)*

$$u(x,t) = v(x - \frac{a^2}{3b}t, t) \exp i(\frac{2a^3}{27b^2}t - \frac{a}{3b}x)$$

is a solution of (1.12) with initial data $u_0(x) = v(x,0) \exp\{-i(a/3b)x\}$*, therefore other solution of (1.12) with two parameters is*

$$u_{\eta,N}(x,t) = g_\eta(b^{-1/3}x + \psi(\eta,N)t) \exp\{ix(b^{-1/3}N - \frac{a}{3b}) + it\phi(\eta,N)\}, \tag{1.16}$$

where $g(x) = (\tilde{\alpha} \cosh x)^{-1}$*,* $\tilde{\alpha} = \alpha/\sqrt{6}$*,* $\phi(\eta,N) = 2a^3/(27b^2) - 3N\eta^2 + N^3 - Na^2b^{-1/3}/(3b)$*,* $\psi(\eta,N) = -a^2b^{-1/3}/(3b) - \eta^2 + 3N^2$ *and*

$$u_{\eta,N}(x,0) = u_{0\eta,N}(x) = g_\eta(b^{-1/3}x) \exp\{ix(b^{-1/3}N - \frac{a}{3b})\}.$$

When $a = 0$ *and* $b = d = 1$ *in (1.12), this solution coincide with the solution obtained in* [20].

3) *If* $e \neq 0$ *and* $b(d+e) > 0$*, then (1.1) have solutions with one parameter:*

$$u_\eta(x,t) = g_\eta(x + \psi(\eta,w)t) \exp i\{wx + \phi(\eta,w)t\},$$

where $w = (c - 2aA^2)/(2e)$*,* $g_\eta(x) = \eta g(\eta x)$*,* $g(x) = (A\cosh x)^{-1}$*,* $A = \sqrt{(e+d)/(6b)}$*,* ψ *and* ϕ *as in (1.13).*

We have also that if u *is a solution of (1.1) then,* $v = \alpha u$ *is a solution of (1.1), where* $\alpha \in \mathbb{C}$*,* $|\alpha| = 1$*, and if* $d \neq e$ *in (1.1) then* $u(x,t) = \exp i\{Cx + Dt + C_0\}$ *is a solution of (1.1), where* $D = aC^2 + bC^3$ *e* $C = c/(e-d)$*.*

Recently there appeared several papers devoted to the global solution of the dispersive type equation, where the framework is based on almost conserved laws and the I-method, see [9, 10, 11, 12, 13]. In this paper we adopt this way in order to obtain our results.

Our aim in this paper is to extend the local solution to a global one. Now, we state our main theorem of global existence:

Theorem 1.2. *The IVP (1.1), with $c = (d-e)a/3b$, is global well-posedness in H^s, $s > 1/4$.*

Notation. The notation to be used is mostly standard. We will use the space-time Lebesgue $L^p_x L^q_T$ endowed with the norm

$$\|f\|_{L^p_x L^q_T} = \big\| \|f\|_{L^q_T} \big\|_{L^p_x} = \Big(\int_{\mathbb{R}} \Big(\int_0^T |f(x,t)|^q dt \Big)^{p/q} dx \Big)^{1/p}.$$

We will use the notation $\|f\|_{L^p_x L^q_t}$ when the integration in the time variable is on the whole real line. In order to define the $X_{s,\beta}$ spaces we consider the following IVP

$$\begin{cases} u_t + iau_{xx} + bu_{xxx} = 0, \quad x,\, t \in \mathbb{R},\; b \neq 0, \\ u(0) = u_0, \end{cases}$$

whose solution is given by $u(x,t) = U(t)u_0(x)$, where the unitary group $U(t)$ is defined as

$$\widehat{U(t)u_0}(\xi) = e^{it(b\xi^3 + a\xi^2)} \widehat{u_0}(\xi).$$

For $s, \beta \in \mathbb{R}$, $X_{s,\beta}$ denotes the completion of the Schwartz space $S(\mathbb{R}^2)$ with respect to the norm

$$\begin{aligned} \|u\|_{s,\beta} \equiv \|u\|_{X_{s,\beta}} \equiv \|U(-t)u\|_{H_{s,\beta}} &\equiv \|\langle\tau\rangle^{\beta} \langle\xi\rangle^s \widehat{U(-t)u}(\xi,\tau)\|_{L^2_\tau L^2_\xi} \\ &= \|\langle \tau - (b\xi^3 + a\xi^2)\rangle^{\beta} \langle\xi\rangle^s \widehat{u}(\xi,\tau)\|_{L^2_\tau L^2_\xi}, \end{aligned}$$

where

$$\widehat{u}(\xi,\tau) \equiv \int_{\mathbb{R}^2} e^{-i(x\xi + t\tau)} u(x,t)\, dx dt.$$

For any time interval $[0,\rho]$, we define the space $X^{\rho}_{s,b}$ by the norm

$$\|u\|_{X^{\rho}_{s,b}} = \inf\{ \|U\|_{X_{s,b}} : U|_{[0,\rho]\times\mathbb{R}} = u \}.$$

The notation $A \lesssim B$ means there exist a constant C such that $A \leq C\, B$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. The notations ξ_{ij} means $\xi_i + \xi_j$, ξ_{ijk} means $\xi_i + \xi_j + \xi_k$, etc. Also we use the notation $m(\xi_i) := m_i$, $m(\xi_{ij}) := m_{ij}$, etc.

The notations for multilinear expressions is the same as in [9, 10], we define a spatial n-multiplier to be any function $M_n(\xi_1, \ldots, \xi_n)$ on the hyperplane

$$\Gamma_n := \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n; \xi_1 + \cdots + \xi_n = 0\},$$

which we endow with the dirac measure $\delta(\xi_1 + \cdots + \xi_n)$. We define the n-linear functional as

$$\Lambda(M_n; f_1, \ldots, f_n) := \int_{\Gamma_n} M_n(\xi_1, \ldots, \xi_n) \prod_1^n \widehat{f}_j(\xi_j),$$

where $f_1, \ldots, f_n$ are complex functions on $\mathbb{R}$. We shall denote

$$\Lambda(M_n; f) := \Lambda(M_n; f, \overline{f}, f, \overline{f} \ldots, f, \overline{f}).$$

For $1 \le j \le n$, $k \ge 1$ we define the elongation $\mathbf{X}_j^k(M_n)$ of M_n to be the multiplier of order $n+k$ given by

$$\mathbf{X}_j^k(M_n)(\xi_1,\dots,\xi_{n+k}) := M_n(\xi_1,\dots,\xi_{j-1},\xi_j,\dots,\xi_{j+k},\xi_{j+k+1},\dots,\xi_{n+k}).$$

2. Almost Conservations Laws

From (1.1) we have

$$\begin{aligned}\partial_t w + ia\,\partial_x^2 w + b\,\partial_x^3 w + ic\,w\bar{w}w + d\,(\partial_x w)\bar{w}w + e\,w(\partial_x \bar{w})w =&0,\\ \partial_t \bar{w} - ia\,\partial_x^2 \bar{w} + b\,\partial_x^3 \bar{w} - ic\,\bar{w}w\bar{w} + d\,(\partial_x \bar{w})w\bar{w} + e\,\bar{w}(\partial_x w)\bar{w} =&0.\end{aligned}$$

Taking Fourier transformation in the above equalities we obtain the following result

Proposition 2.1. *Let $n \ge 2$, be an even integer, and let M_n be a multiplier of order n, then*

$$\begin{aligned}\partial_t\Lambda_n(M_n;w) =& i\Lambda_n(M_n\Upsilon_n^{a,b};w) - i\Lambda_{n+2}\left(\sum_{j=1}^{n}\Upsilon_{j,n+2}^{c,e}\boldsymbol{X}_j^2(M_n);w\right)\\ &- id\Lambda_{n+2}\left(\sum_{j=1}^{n/2}\boldsymbol{X}_{2j-1}^2(M_n)\xi_{2j-1} + \sum_{j=1}^{n/2}\boldsymbol{X}_{2j}^2(M_n)\xi_{2j+2};w\right),\end{aligned}\tag{2.1}$$

where $\Upsilon_n^{a,b} = \sum_{j=1}^n((-1)^{j-1}a\xi_j^2 + b\xi_j^3)$ and $\Upsilon_{j,n+2}^{c,e} = (-1)^{j-1}c + e\xi_{j+1}$.

We define the first modified energy as

$$E_1 = k_1\Lambda_2(M_2;w), \quad M_2(\xi_1,\xi_2) = \xi_1\xi_2 m(\xi_1)m(\xi_2),\tag{2.2}$$

where $k_1 = 3be$, and the second modified energy as

$$E_2 = E_1 + \Lambda_4(\delta_4),\tag{2.3}$$

where the 4-multiplier δ_4 will be choosed after. By (2.1) we get

$$\begin{aligned}\partial_t E_2 =& \partial_t E_1 + \partial_t\Lambda_4(\delta_4) = k_1\Lambda_2(M_2\Upsilon_2^{a,b}) - ik_1\Lambda_4\left(\sum_{j=1}^{2}\Upsilon_{j,4}^{c,e}\mathbf{X}_j^2(M_2)\right)\\ &- idk_1\Lambda_4(\mathbf{X}_1^2(M_2)\xi_1 + \mathbf{X}_2^2(M_2)\xi_4) + i\Lambda_4(\delta_4\Upsilon_4^{a,b}) - i\Lambda_6\left(\sum_{j=1}^{4}\Upsilon_{j,6}^{c,e}\mathbf{X}_j^2(\delta_4)\right)\\ &- id\Lambda_6\left(\sum_{j=1}^{2}\mathbf{X}_{2j-1}^2(\delta_4)\xi_{2j-1} + \sum_{j=1}^{2}\mathbf{X}_{2j}^2(\delta_4)\xi_{2j+2}\right),\end{aligned}\tag{2.4}$$

it is clear that $\Lambda_2(M_2\Upsilon_2^{a,b}) = 0$. Now if $\tilde{M}_n$ is an n- multiplier $\Lambda_n(\tilde{M}_n)$ is invariant under permutations of the even ξ_j indices or of the odd ξ_j indices, therefore for achieve a cancellation of the 4-linear expression, we choose δ_4 such that

$$\Upsilon_4^{a,b}\delta_4 = \frac{ck_1}{2}(\xi_1^2 m_1^2 - \xi_2^2 m_2^2 + \xi_3^2 m_3^2 - \xi_4^2 m_4^2) - \frac{ek_1}{2}(\xi_2\xi_4^2 m_4^2 + \xi_4\xi_2^2 m_2^2 + \xi_1\xi_3^2 m_3^2 + \xi_3\xi_1^2 m_1^2) - \frac{dk_1}{2}(\xi_1\xi_4^2 m_4^2 + \xi_4\xi_1^2 m_1^2 + \xi_3\xi_2^2 m_2^2 + \xi_2\xi_3^2 m_3^2), \tag{2.5}$$

consequently from (2.4) we get

$$\partial_t E_2 = \Lambda_6(\delta_6), \tag{2.6}$$

with

$$\delta_6 = \frac{-ie}{36}\sum_{\substack{\{k,m,o\}=\{1,3,5\}\\ \{l,n,p\}=\{2,4,6\}}}[\xi_l\delta_4(\xi_{klm},\xi_n,\xi_o,\xi_p) + \xi_m\delta_4(\xi_k,\xi_{lmn},\xi_o,\xi_p) + \xi_n\delta_4(\xi_k,\xi_l,\xi_{mno},\xi_p) + \xi_o\delta_4(\xi_k,\xi_l,\xi_m,\xi_{nop})] - \frac{id}{36}\sum_{\substack{\{k,m,o\}=\{1,3,5\}\\ \{l,n,p\}=\{2,4,6\}}}[\xi_k\delta_4(\xi_{klm},\xi_n,\xi_o,\xi_p) + \xi_m\delta_4(\xi_k,\xi_l,\xi_{mno},\xi_p) + \xi_n\delta_4(\xi_k,\xi_{lmn},\xi_o,\xi_p) + \xi_p\delta_4(\xi_k,\xi_l,\xi_m,\xi_{nop})].$$

Proposition 2.2. *If $m(\xi) = 1$ for all ξ, then*

$$\partial_t E_2 = 0.$$

Proof. From definition of E_2, we have

$$E_2 = 3be\Lambda_2(\xi_1\xi_2; w) + \Lambda_4(\delta_4; w), \tag{2.7}$$

where

$$\Upsilon_4^{a,b}\delta_4 = \frac{ck_1}{2}(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2) - \frac{ek_1}{2}(\xi_2\xi_4^2 + \xi_4\xi_2^2 + \xi_1\xi_3^2 + \xi_3\xi_1^2) - \frac{dk_1}{2}(\xi_1\xi_4^2 + \xi_4\xi_1^2 + \xi_3\xi_2^2 + \xi_2\xi_3^2).$$

If $\xi_1 + \cdots + \xi_4 = 0$, then $\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 = 2\xi_{12}\xi_{14}$ and $\xi_1^3 + \cdots + \xi_4^3 = 3\xi_{12}\xi_{13}\xi_{14}$, therefore

$$\Upsilon_4^{a,b} = 2a\xi_{12}\xi_{14} + 3b\xi_{12}\xi_{13}\xi_{14} = \xi_{12}\xi_{14}(2a + 3b\xi_{13}). \tag{2.8}$$

On the other hand $\xi_2\xi_4^2 + \xi_4\xi_2^2 + \xi_1\xi_3^2 + \xi_3\xi_1^2 = -\xi_{12}\xi_{13}\xi_{14}$ (see Lemma 3.5 in [9] and Remark 3.6 in [10]), similarly $\xi_1\xi_4^2 + \xi_4\xi_1^2 + \xi_3\xi_2^2 + \xi_2\xi_3^2 = -\xi_{12}\xi_{13}\xi_{14}$, hence

$$\delta_4 = \frac{3be}{2}\frac{2c\xi_{12}\xi_{14} + (d+e)\xi_{12}\xi_{13}\xi_{14}}{\xi_{12}\xi_{14}(2a + 3b\xi_{13})} = \frac{e(d+e)}{2}.$$

And from (2.7) we get

$$
\begin{aligned}
E_2 =& 3be\Lambda_2(\xi_1\xi_2;w) + \frac{e(d+e)}{2}\Lambda_4(1;w)\\
=& -3be\int_{\mathbb{R}}|w_x|^2 + \frac{e(d+e)}{2}\int_{\mathbb{R}}|w|^4\\
=& -I_2(w).
\end{aligned}
$$

This conclude the proof of the proposition. □

In the following sections we will consider $a=c=0$ in the IVP (1.1) (see (1.4) and (1.5)).

3. Preliminary results

For the estimates on the multipliers we use the following elementary results.

Lemma 3.1. **1)** *(Double mean value theorem DMVT)*
Let $f\in C^2(\mathbb{R})$, and $\max\{|\eta|,|\lambda|\}\ll\xi$, then

$$|f(\xi+\eta+\lambda)-f(\xi+\eta)-f(\xi+\lambda)+f(\xi)|\lesssim|f''(\theta)||\eta||\lambda|,$$

where $|\theta|\sim|\xi|$.

2) *(Triple mean value theorem TMVT)*
Let $f\in C^3(\mathbb{R})$, and $\max\{|\eta|,|\lambda|,|\gamma|\}\ll\xi$, then

$$|f(\xi+\eta+\lambda+\gamma)-f(\xi+\lambda+\eta)-f(\xi+\eta+\gamma)-f(\xi+\lambda+\gamma)+f(\xi+\eta)$$

$$+f(\xi+\lambda)+f(\xi+\gamma)-f(\xi)|\lesssim|f'''(\theta)||\eta||\lambda||\gamma|,$$

where $|\theta|\sim|\xi|$.

And for the proof of Proposition 5.2, shall be fundamental the improved Strichartz estimate.

Lemma 3.2. *Let $s>1/4$, $v_1,v_2\in\mathbf{S}(\mathbb{R}\times\mathbb{R})$ such that supp $\widehat{v_1}\subset\{|\xi|\sim N\}$ and supp $\widehat{v_2}\subset\{|\xi|\ll N\}$, then*

$$\|v_1v_2\|_{L^4_xL^2_t}\lesssim\frac{1}{(1-4s)^{1/4}}\frac{1}{N}\|v_2\|_{X^\rho_{s,1/2+}}\|v_1\|_{X^\rho_{0,1/2+}}.$$

Proof. As in [9] is sufficient to prove

$$\|v_1v_2\|_{L^4_xL^2_t}\lesssim\frac{1}{(1-4s)^{1/4}}\frac{1}{N}\|\phi\|_{H^s}\|\psi\|_{L^2},$$

where $v_1 = U(t)\psi$ and $v_2 = U(t)\phi$. By duality, definition of v_1, v_2, Fubini theorem and Plancherel identity in the spatial variable we have

$$\|v_1v_2\|_{L^4_xL^2_t} \lesssim \sup_{\|F\|_{L^{4/3}_xL^2_t}\leq 1} \int_{\mathbb{R}^2} |\widehat{\phi}(y)\widehat{\psi}(z)\widehat{F}(z+y, z^3+y^3)|dzdy$$
$$=\frac{1}{N^2} \sup_{\|F\|_{L^{4/3}_xL^2_t}\leq 1} \int_{\mathbb{R}^2} |\widehat{\phi}(y(s,r))\widehat{\psi}(z(s,r))\widehat{F}(s,r)|drds,$$

where we used the change of variable $z + y = s$, $z^3 + y^3 = r$, which has Jacobian of size N^2. Now if we applying Hölder inequality and a change of variables back for z and y, we obtain

$$\|v_1v_2\|_{L^4_xL^2_t} \lesssim\frac{1}{N^2} \sup_{\|F\|_{L^{4/3}_xL^2_t}\leq 1} \|\widehat{\phi}(y(s,r))\widehat{\psi}(z(s,r))\|_{L^{4/3}_sL^2_r}\|\widehat{F}\|_{L^4_xL^2_t}$$
$$\leq\frac{1}{N} \sup_{\|F\|_{L^{4/3}_xL^2_t}\leq 1} \|\psi\|_{L^2_z}\|\widehat{\phi}\|_{L^{4/3}_y}\|\widehat{F}\|_{L^4_xL^2_t},$$

where the Fourier transform of F is taking only in the space variable. Using Hölder inequality we obtain for $s > 1/4$

$$\int_{\mathbb{R}} |\widehat{\phi}|^{4/3} \leq \Big(\int_{\mathbb{R}} \langle\xi^2\rangle^s|\widehat{\phi}|^2\Big)^{2/3}\Big(\int_{\mathbb{R}} \frac{1}{\langle\xi^2\rangle^{2s}}\Big)^{1/3},$$

therefore

$$\|\widehat{\phi}\|_{L^{4/3}_y} \leq \frac{1}{(1-4s)^{1/4}}\|\phi\|_{H^s},$$

and by Hausdorff-Young inequality and Minkowsky integral inequality, we get

$$\|\widehat{F}\|_{L^4_xL^2_t} \leq \|\widehat{F}\|_{L^2_tL^4_x} \leq \|F\|_{L^2_tL^{4/3}_x} \leq \|F\|_{L^{4/3}_xL^2_t} \leq 1.$$

This completes the proof. □

We define the Fourier multiplier operator I with symbol

$$m(\xi) = \begin{cases} 1, & |\xi| < N, \\ \frac{N^{1-s}}{|\xi|^{1-s}}, & |\xi| > 2N. \end{cases} \tag{3.1}$$

We have $I : H^s \mapsto H^1$. For the local result we define the Fourier multiplier operator L, with symbol

$$l(\xi) = m(\xi)\langle\xi\rangle^{1-s} = \begin{cases} \langle\xi\rangle^{1-s}, & |\xi| < N, \\ \langle\xi\rangle^{1-s}\frac{N^{1-s}}{|\xi|^{1-s}}, & |\xi| > 2N. \end{cases}$$

Is obvious that

$$\|Iu\|_{H^1} = \|Lu\|_{H^s}, \quad \|Iu\|_{X_{1,b}} = \|Lu\|_{X_{s,b}}, \tag{3.2}$$

and for $s \in [0, 1)$ is $1 \leq l(\xi) \lesssim N^{1-s}$, therefore

$$\|u\|_{s',b'} \lesssim \|Iu\|_{s'-s+1,b'} \lesssim N^{1-s}\|u\|_{s',b'}, \quad s \in [0, 1),$$

observe that if $V|_{[0,\rho]\times\mathbb{R}} = Iu$, $V \in X_{s'-s+1,b'}$, then U defined by $\widehat{U} = (1/m)\widehat{V}$, satisfies

$$\|U\|_{s',b'} \lesssim \|V\|_{s'-s+1,b'},$$

moreover in $[0,\rho]$ is $U|_{[0,\rho]\times\mathbb{R}} = u$, therefore

$$\|u\|_{X^{\rho}_{s',b'}} \lesssim \|Iu\|_{X^{\rho}_{s'-s+1,b'}}. \tag{3.3}$$

Also we have

$$l(\xi_1+\xi_2) \lesssim l(\xi_1) + l(\xi_2). \tag{3.4}$$

In fact, for see this, without lost of generality we can assume $|\xi_1| \geq |\xi_2|$, we consider two cases:

i) If $|\xi_1| \leq N$, then we have $|\xi_1+\xi_2| \leq 2N$, this implies

$$l(\xi_1+\xi_2) \sim \langle \xi_1+\xi_2\rangle^{1-s} \leq \langle \xi_1\rangle^{1-s} + \langle \xi_2\rangle^{1-s} = l(\xi_1) + l(\xi_2).$$

ii) If $|\xi_1| \geq N$, then we have $l(\xi_1) \sim N^{1-s}$, thus for all ξ, $l(\xi) \lesssim l(\xi_1)$, in particular $l(\xi_1+\xi_2) \lesssim l(\xi_1) \leq l(\xi_1) + l(\xi_2)$. Note that (3.4) implies $l(\xi_1+\xi_2) \lesssim l(\xi_1)l(\xi_2)$.

In the proof of Theorem 1.2 we will use the following local result.

Theorem 3.3. *Let $s \geq 1/4$, then the IVP (1.1) is locally well-posed for data φ, with $I\varphi \in H^1$ where the time of existence satisfies*

$$\delta \sim \|I\varphi\|_{H^1}^{-\theta}, \tag{3.5}$$

with $\theta > 0$. Moreover the solution of the IVP (1.1), is such that

$$\|Iu\|_{X^{\delta}_{1,1/2+}} \lesssim \|Iu_0\|_{H^1}. \tag{3.6}$$

Proof. The Theorem 3.3 is practically done in [29] (see also [32]), in fact, is sufficient to prove

$$\|L(uv\overline{w}_x)\|_{X_{s,-1/2+}} \lesssim \|Lu\|_{X_{s,1/2+}}\|Lv\|_{X_{s,1/2+}}\|Lw\|_{X_{s,1/2+}}, \tag{3.7}$$

$$\|L(u\overline{v}w_x)\|_{X_{s,-1/2+}} \lesssim |Lu\|_{X_{s,1/2+}}\|Iv\|_{X_{s,1/2+}}\|Lw\|_{X_{s,1/2+}},$$

$$\|L(u\overline{v}w)\|_{X_{s,-1/2+}} \lesssim \|Lu\|_{X_{s,1/2+}}\|Lv\|_{X_{s,1/2+}}\|Lw\|_{X_{s,1/2+}}, \tag{3.8}$$

in order to prove the first inequality we make the following decomposition

$$\begin{aligned} l(\xi)\widehat{uv\overline{w}_x}(\xi,\tau) =& l(\xi)\int_{|\xi_1|>2N}\zeta + l(\xi)\int_{\substack{|\xi_1|\leq 2N\\ |\xi_2|>2N}}\zeta + l(\xi)\int_{\substack{|\xi_1|\leq 2N\\ |\xi_2|\leq 2N\\ |\xi-\xi_1-\xi_2|>2N}}\zeta \\ &+ l(\xi)\int_{\substack{|\xi_1|\leq 2N\\ |\xi_2|\leq 2N\\ |\xi-\xi_1-\xi_2|\leq 2N}}\zeta, \end{aligned}$$

where $\zeta := \widehat{u}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)\widehat{v}(\xi_1,\tau_1)\widehat{\overline{w}}_x(\xi_2,\tau_2)$, thus

$$|l(\xi)\widehat{uv\overline{w}_x}(\xi,\tau)| \lesssim |\widehat{uv_1\widehat{\overline{w}}_x}| + |\widehat{uv_2\widehat{\overline{v_3}}_x}| + |\widehat{v_2\widehat{\overline{v_4}}_x v_5}| + \langle\xi\rangle^{1-s}|\widehat{v_2\widehat{\overline{v_4}}_x v_6}|,$$

with

$$\begin{aligned}
&\widehat{v_1}(\xi,\tau) = \chi_{|\xi|>2N}\widehat{v}(\xi,\tau)l(\xi), \quad \widehat{v_2}(\xi,\tau) = \chi_{|\xi|\le 2N}\widehat{v}(\xi,\tau),\\
&\widehat{\overline{v_3}}_x(\xi,\tau) = \chi_{|\xi|>2N}\widehat{\overline{w}}_x(\xi,\tau)l(\xi), \quad \widehat{\overline{v_4}}_x(\xi,\tau) = \chi_{|\xi|\le 2N}\widehat{\overline{w}}_x(\xi,\tau),\\
&\widehat{v_5}(\xi,\tau) = \chi_{|\xi|>2N}\widehat{u}(\xi,\tau)l(\xi), \quad \widehat{v_6}(\xi,\tau) = \chi_{|\xi|\le 2N}\widehat{u}(\xi,\tau),
\end{aligned}$$

and applying Proposition 2.7 in [29] (see also Theorem 2.1 in [32]) we obtain (3.7). For the other inequalities we make an analogous decomposition. □

We also have a result of local well-posed with the interval of existence as in (3.5), without the use of the theory of the spaces $X_{s,b}$.

In fact, let

$$\begin{aligned}
|||u|||_{\Delta T,s} =& \|\partial_x u\|_{L^\infty_x L^2_{\Delta T}} + \|D^s_x\partial_x u\|_{L^\infty_x L^2_{\Delta T}} + \|D^{s-1/4}_x\partial_x u\|_{L^{20}_x L^{5/2}_{\Delta T}} + \|u\|_{L^5_x L^{10}_{\Delta T}}\\
&+ \|D^s_x u\|_{L^5_x L^{10}_{\Delta T}} + \|u\|_{L^4_x L^\infty_{\Delta T}} + \|u\|_{L^8_x L^8_{\Delta T}} + \|D^s_x u\|_{L^8_x L^8_{\Delta T}}.
\end{aligned}$$

Theorem 3.4. *Let $s \ge 1/4$, and $a, b \in \mathbb{R}, b \neq 0$, $c, d, e \in \mathbb{C}$, then the IVP (1.1) is locally well-posed for data φ, with $I\varphi \in H^1$. Moreover the solution is such that*

$$|||u|||_{\delta,s} \lesssim \|I\varphi\|_{H^1},$$

where δ satisfies (3.5) with $\theta = 4$.

Proof. The theorem follows from the proof in [27] if we prove

$$|||U(t)u_0|||_{\delta,s} \lesssim \|Iu_0\|_{H^1} = \|Lu_0\|_{H^s},$$

we consider the first term in the definition of $|||\cdot|||_T$, we will prove

$$\|\partial_x U(t)u_0\|_{L^\infty_x L^2_\delta} \lesssim \|Lu_0\|_{L^2}. \tag{3.9}$$

The inequality (3.9) is equivalent with

$$\|L^{-1}\partial_x U(t)u_0\|_{L^\infty_x L^2_\delta} \lesssim \|u_0\|_{L^2},$$

where the Fourier multiplier operator L^{-1} have symbol $1/l(\xi) \le 1$, it is easy to see that

$$\|L^{-1}\partial_x U(t)u_0\|_{L^\infty_x L^2_\delta} \le \|\partial_x U(t)L^{-1}u_0\|_{L^\infty_x L^2_\delta} \le \|L^{-1}u_0\|_{L^2} \le \|u_0\|_{L^2}. \tag{3.10}$$

We proceed similarly with the others terms. □

Lemma 3.5. *For any $s_1 \geq 1/4$, $s_2 \geq 0$ and $b > 1/2$ we have*

$$\|u\|_{L_x^4 L_\delta^\infty} \lesssim \|u\|_{X^\rho_{s_1,b}}, \tag{3.11}$$

$$\|u\|_{L_x^8 L_\delta^8} \lesssim \|u\|_{X^\rho_{s_2,b}}, \tag{3.12}$$

$$\|u\|_{L_x^6 L_\delta^6} \lesssim \|u\|_{X^\rho_{s_2,b}}. \tag{3.13}$$

Proof. The inequalities (3.11) and (3.12), follows from

$$\|U(t)u_0\|_{L_x^4 L_\delta^\infty} \lesssim \|u_0\|_{H^{1/4}}, \quad \|U(t)u_0\|_{L_x^8 L_\delta^8} \lesssim \|u_0\|_{L^2}$$

and from a standard argument, see for example [2, 10, 14].

The inequality (3.13) follows by interpolation between $\|v\|_{L_x^8 L_\delta^8} \lesssim \|v\|_{X_{0,1/2+}}$ and the trivial estimate $\|v\|_{L_x^2 L_\delta^2} \leq \|v\|_{0,0}$. □

Remark 3.6. *Actually the inequality (3.8) is valid for all $s > -1/4$ (see [5]).*

4. Estimates for δ_4 and δ_6

From here onwards we will consider the notation $|\xi_i| = N_i$, $m(N_i) = m_i$, $|\xi_{ij}| = N_{ij}$, $m(N_{ij}) = m_{ij}$, etc. Given ford number N_1, N_2, N_3, N_4 and $\mathcal{C} = \{N_1, \dots, N_4\}$, we will note $N_s = \max \mathcal{C}$, $N_a = \max \mathcal{C} \setminus \{N_s\}$, $N_t = \max \mathcal{C} \setminus \{N_s, N_a\}$, $N_b = \min \mathcal{C}$, in this way

$$N_s \geq N_a \geq N_t \geq N_b.$$

Proposition 4.1. *Let m defined as in (3.1), then*

$$|\delta_4| \lesssim m^2(N_s) \tag{4.1}$$

and

$$|\delta_6| \lesssim N_s m^2(N_s). \tag{4.2}$$

In order to prove (4.1) we will use the following proposition in similar form like when $m = 1$.

Proposition 4.2. *Let m defined as in (3.1), then*

$$|\xi_2\xi_4^2m_4^2 + \xi_4\xi_2^2m_2^2 + \xi_1\xi_3^2m_3^2 + \xi_3\xi_1^2m_1^2| \lesssim m^2(N_s)|\xi_{12}\xi_{13}\xi_{14}|, \tag{4.3}$$

and

$$|\xi_1\xi_4^2m_4^2 + \xi_4\xi_1^2m_1^2 + \xi_2\xi_3^2m_3^2 + \xi_3\xi_2^2m_2^2| \lesssim m^2(N_s)|\xi_{12}\xi_{13}\xi_{14}|. \tag{4.4}$$

Proof. Without lost of generality we can assume $|\xi_1| = N_s$, and by symmetry $|\xi_{12}| \leq |\xi_{14}|$. In [10] (Lemma 4.1) they proved that

$$|\xi_2\xi_4^2m_4^2 + \xi_4\xi_2^2m_2^2 + \xi_1\xi_3^2m_3^2 + \xi_3\xi_1^2m_1^2| \lesssim m^2(N_s)|\xi_{12}\xi_{14}|N_s,$$

therefore we can suppose $|\xi_{13}| \ll N_s$, this implies $|\xi_1| \sim |\xi_3|$. Let $f(\xi) = \xi m(\xi)$, observing that $\xi_{12}\xi_{14} = \xi_2\xi_4 - \xi_1\xi_3$, we have

$$\begin{aligned}\xi_2\xi_4^2m_4^2 + \xi_4\xi_2^2m_2^2 + \xi_1\xi_3^2m_3^2 + \xi_3\xi_1^2m_1^2 &= \xi_2\xi_4(f(\xi_2) + f(\xi_4)) + \xi_1\xi_3(f(\xi_1) + f(\xi_3)) \\ &= \xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4)) - \xi_{12}\xi_{14}(f(\xi_1) + f(\xi_3)). \quad (4.5)\end{aligned}$$

In the second term of (4.5) we can use the medium value theorem (MVT) for to obtain

$$|\xi_{12}\xi_{14}(f(\xi_1) + f(\xi_3))| = |\xi_{12}\xi_{14}(f(\xi_1) - f(-\xi_3))| \lesssim |\xi_{12}\xi_{14}\xi_{13}|m^2(N_s),$$

where we used that $|\xi_{13}| \ll N_s$, and $|f'(\xi_1)| \sim |m^2(\xi_1)|$. Therefore we will only estimate the first term in (4.5).

We consider two cases:

1) $|\xi_{14}| \gtrsim |\xi_3|$, in this case we consider two sub-cases

a) If $|\xi_{12}| \ll |\xi_1|$, then using the DMVT (Lemma 3.1) with $\xi = -\xi_1$, $\lambda = \xi_{12}$ and $\eta = \xi_{13}$

$$|\xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4))| \lesssim |\xi_{14}|N_s|\xi_{12}\xi_{13}|\frac{m^2(N_s)}{N_s},$$

where we also used that $|\xi_2| \le |\xi_1| \sim |\xi_3| \lesssim |\xi_{14}|$ and $|f''(\xi_1)| \lesssim m^2(\xi_1)/|\xi_1|$.

b) If $|\xi_{12}| \gtrsim |\xi_1|$, here we proceed similarly as in [10] (Lemma 4.1). Using the fact that $N_s \lesssim |\xi_{12}| \le |\xi_{14}|$, $(m^2(\xi)\xi^2)' \sim m^2(\xi)\xi$, $m^2(\xi)\xi$ is nondecreasing and the MVT we have

$$\begin{aligned}|\xi_2\xi_4^2m_4^2 + \xi_4\xi_2^2m_2^2 + \xi_1\xi_3^2m_3^2 + \xi_3\xi_1^2m_1^2| &= |\xi_3(m_1^2\xi_1^2 - m_{1-13}^2\xi_{1-13}^2) + \xi_2(m_4^2\xi_4^2 \\ &\quad - m_{4+13}^2\xi_{4+13}^2) - \xi_{24}(m_3^2\xi_3^2 - m_{4+13}^2\xi_{4+13}^2)| \lesssim |\xi_{13}|N_s^2m^2(N_s).\end{aligned}$$

Hence we obtain (4.3) in this sub-case.

2) $|\xi_{14}| \ll |\xi_3|$, using the TMVT considering in Lemma 3.1: $\xi = -\xi_1$, $\lambda = \xi_{12}$, $\eta = \xi_{13}$ and $\gamma = \xi_{14}$ we have

$$|\xi_2\xi_4(f(\xi_1) + f(\xi_2) + f(\xi_3) + f(\xi_4))| \lesssim N_s^2|\xi_{12}\xi_{13}\xi_{14}|\frac{m^2(N_s)}{N_s^2},$$

where we also used that $|f'''(\xi_1)| \lesssim m^2(\xi_1)/|\xi_1|^2$.

Now, in order to obtain (4.4), using (4.3) we get

$$|\xi_1\xi_4^2m_4^2 + \xi_4\xi_1^2m_1^2 + \xi_2\xi_3^2m_3^2 + \xi_3\xi_2^2m_2^2| \lesssim m^2(N_s)|\xi_{21}\xi_{23}\xi_{24}| = m^2(N_s)|\xi_{12}\xi_{14}\xi_{13}|.$$

This completes the proof. □

By (2.5), (2.8) and Proposition 4.2, we have (4.1). The estimate (4.2) is obvious.

5. Estimates 4-lineal and 6-lineal

The following lemma will be used frequently in the estimates 4-lineal and 6-lineal.

Lemma 5.1. *Let $n \geq 2$ a even integer, $w_1, \dots, w_n \in \mathbf{S}(\mathbb{R})$, then*

$$\int_{\xi_1+\dots+\xi_n=0} \widehat{w}_1 \widehat{\overline{w}}_2 \dots \widehat{w}_{n-1} \widehat{\overline{w}}_n = \int_{\mathbb{R}} w_1 \overline{w}_2 \dots w_{n-1} \overline{w}_n. \tag{5.1}$$

In the proof of our global result, we will need the following properties.

Proposition 5.2. *Let $w \in \mathbf{S}(\mathbb{R}\times\mathbb{R})$, then we have*

$$\left| \int_0^{\rho} \Lambda_6(\delta_6; w(t))dt \right| \lesssim N^{-3} \|Iw\|^6_{X^{\rho}_{1,1/2+}} \tag{5.2}$$

and

$$|\Lambda_4(\delta_4; w(t))| \lesssim \|Iw\|^4_{H^1}. \tag{5.3}$$

Proof. As in [9, 10, 11], we first perform a Littlewood-Paley decomposition of the six factors w, so that the ξ_i are essentially the constants N_i, $i = 1, \dots, 6$. To recover the sum at the end we borrow a $N_s^{-\epsilon}$ from the large denominator N_s and often this will not be mentioned. Also without loss of generality we can assume that the Fourier transforms in the left-side of (5.2) and (5.3) are real and nonnegative.

Let $I = \{s, a, t, b\}$ the set of indices such that $N_s \geq N_a \geq N_t \geq N_b$. We will proved first (5.2), we divide the proof into two cases.
1) $N_b \gtrsim N$, by definition of m we have $N_t m_t \gtrsim N$ and $N_b m_b \gtrsim N$, therefore

$$N_s m_s^2 \lesssim N^{-3} N_s m_s N_a m_a N_t m_t N_b m_b,$$

and consequently by (5.1), Hölder inequality, (3.3) and Lemma 3.5, we have

$$\begin{aligned}
\left| \int_0^{\rho} \Lambda_6(\delta_6; w(t))dt \right| &\lesssim N^{-3} \int_0^{\rho} \int_{\mathbb{R}} \prod_{j\in I} D_x I w_j \prod_{j\notin I} w_j dx\, dt \\
&\lesssim N^{-3} \prod_{j\in I} \|D_x I w_j\|_{L^6_x L^6_\rho} \prod_{j\notin I} \|w_j\|_{L^6_x L^6_\rho} \\
&\lesssim N^{-3} \prod_{j\in I} \|I w_j\|_{X^{\rho}_{1,1/2+}} \prod_{j\notin I} \|I w_j\|_{X^{\rho}_{1-s,1/2+}} \\
&\lesssim N^{-3} \|Iw\|^6_{X^{\rho}_{1,1/2+}}.
\end{aligned}$$

2) $N_b \ll N$, by (2.6) and Proposition 2.2, if $N_s \ll N$, then $\Lambda_6(\delta_6) = 0$, therefore we can assume $N_s \gtrsim N$, and for $\xi_1 + \dots + \xi_6 = 0$ this implies $N_s \sim N_a \gtrsim N$, hence

$$N_s m_s^2 \lesssim N^{-1} N_s m_s N_a m_a,$$

by (5.1), Hölder inequality, (3.3) and Lemmas 3.5 and 3.2 one obtains

$$\begin{aligned}\left|\int_0^\rho \Lambda_6(\delta_6; w(t))dt\right| &\lesssim N^{-1}\int_0^\rho\int_{\mathbb{R}} D_xIw_sD_xIw_a \prod_{j\notin\{s,a\}} w_j dx\, dt\\ &\lesssim N^{-1}\|(D_xIw_s)w_b\|_{L^4_xL^2_\rho}\|(D_xIw_a)w_p\|_{L^4_xL^2_\rho}\|w_t\|_{L^4_xL^\infty_\rho}\|w_q\|_{L^4_xL^\infty_\rho}\\ &\lesssim N^{-3}\|Iw_s\|_{X^\rho_{1,1/2+}}\|w_b\|_{X^\rho_{s_0,1/2+}}\|Iw_a\|_{X^\rho_{1,1/2+}}\|w_p\|_{X^\rho_{s_0,1/2+}}\\ &\quad \|w_t\|_{X^\rho_{1/4,1/2+}}\|w_q\|_{X^\rho_{1/4,1/2+}}\\ &\lesssim N^{-3}\|Iw\|^6_{X^\rho_{1,1/2+}},\end{aligned}$$

where $1/4 < s_0 < 1$.

For to prove (5.3), by (4.1) and (5.1) we have

$$\begin{aligned}|\Lambda_4(\delta_4; w(t))| &\lesssim \int_{\xi_1+\cdots+\xi_n} \delta_4(\xi_1,\ldots,\xi_n)\widehat{w}_1\widehat{\overline{w}}_2\widehat{w}_3\widehat{\overline{w}}_4\\ &\lesssim \int_{\mathbb{R}} |w(t)|^4dx \lesssim \|w(t)\|^4_{H^{1/4}}\\ &\lesssim \|Iw\|^4_{H^1}.\end{aligned}$$

Which finished the proof. □

6. Proof of Theorem 1.2

We will use the following results.

Lemma 6.1. *If u is a solution of IVP (1.1), then*

$$\|Iu(t)\|_{L^2} \le \|I\varphi\|_{H^{1-s}}.$$

for $0 \le s < 1$.

Proof. The lemma follows from definition of I, the conservation law in L^2 and definition of $l(\xi)$. □

Lemma 6.2. *If u is a solution of IVP (1.1), then*

$$|E_2(t) - E_1(t)| \le c\|I\varphi\|^4_{H^1} + cE_1(t)^4. \tag{6.1}$$

If k is a positive integer and $u(t)$ is defined in the time interval $[0, k]$, then

$$E_2(k) = E_1(0) + \Lambda_4(\delta_4)(0) + \sum_{j=1}^k \int_{j-1}^j \Lambda_6(\delta_6)(t)\, dt. \tag{6.2}$$

Proof. The inequality (6.1) is obvious from (2.3), (5.3) and Lemma 6.1.

By (2.6) we have

$$E_2(k) = E_2(0) + \sum_{j=1}^k \int_{j-1}^j \Lambda_6(\delta_6)(t)\, dt.$$

and by (2.3) we obtain (6.2). □

6.1. **Rescaling.** We know that if $u(x,t)$ is a solution of (1.5) with initial data $u(x,0)=\varphi$, then

$$u_\lambda(x,t)=\frac{1}{\lambda}u(\frac{x}{\lambda},\frac{t}{\lambda^3}),$$

is also a solution of (1.5) with initial data

$$u_\lambda(x,0)=\frac{1}{\lambda}u(\frac{x}{\lambda},0)=\frac{1}{\lambda}\varphi(\frac{x}{\lambda}):=\varphi_\lambda.$$

Let $c_0\in(0,1)$ a constant to be chosen later, we have

$$\begin{aligned}\|I\varphi_\lambda\|_{H^1}&\sim\|\partial_xI\varphi_\lambda\|_{L^2}+\|I\varphi_\lambda\|_{L^2}\\&\lesssim\frac{N^{1-s}}{\lambda^{1/2+s}}\|D_x^s\varphi\|_{L^2}+\frac{1}{\sqrt{\lambda}}\|\varphi\|_{L^2}\\&<c_0.\end{aligned}$$

taking

$$\lambda\sim N^{\frac{2(1-s)}{1+2s}}\left(\frac{\|D_x^s\varphi\|_{L^2}}{c_0}\right)^{\frac{2}{1+2s}}\quad\text{and}\quad N>\left(\frac{\|\varphi\|_{L^2}}{c_0}\right)^{\frac{2s}{1-s}}. \tag{6.3}$$

6.2. **Iteration.** Without lost of generality we can assume $k_1=1$ in (2.2). We consider our solution rescaled with initial data

$$\|I\varphi\|_{H^1}=\epsilon_0<c_0<1,$$

then by Theorem 3.3 we have a solution of (1.1) defined in the time interval $[0,1]$. For to extend the solution of local theorem in the time interval $[0,\lambda^3T]$ we need to prove that $\|Iu(n)\|_{H^1}\lesssim\epsilon_0$, for all $n\in\{1,2,\dots,m_{\lambda,T}\}=W$, where $m_{\lambda,T}\sim\lambda^3T$. Indeed we will prove that

$$\|Iu(n)\|_{H^1}^2\le 3\epsilon_0^2,\quad n\in W, \tag{6.4}$$

but as $\|Iu(t)\|_{H^1}^2=\|Iu(t)\|_{L^2}+\|\partial_xIu(t)\|_{L^2}$, by Lemma 6.1 is sufficient to prove

$$\|\partial_xIu(n)\|_{L^2}^2\le 2\epsilon_0^2,\quad n\in W. \tag{6.5}$$

We will prove (6.5) by induction.

1) When $k=1$, we suppose by contradiction that $\|\partial_xIu(1)\|_{L^2}^2>2\epsilon_0^2$, then there exist $t_0\in(0,1)$ such that $\|\partial_xIu(t_0)\|_{L^2}^2=2\epsilon_0^2$, from (6.1) we have

$$|E_2(t_0)-2\epsilon_0^2|\le 5c\epsilon_0^4,$$

and using (2.3), (2.6), (3.6) and (5.2) we obtain

$$E_2(t_0)=E_1(0)+\Lambda_4(\delta_4)(0)+\int_0^{t_0}\Lambda_6(\delta_6)(0),$$

and from here

$$|E_2(t_0)-\epsilon_0^2|\le 5c\epsilon_0^4+\frac{1}{N^3}8c\epsilon_0^6,$$

hence if $\epsilon_0^2 < \frac{1}{20c}$, we have

$$\epsilon_0^2 \leq |2\epsilon_0^2 - E_2(t_0)| + |E_2(t_0) - \epsilon_0^2| \leq 5c\epsilon_0^4 + 8c\epsilon_0^6 + 5c\epsilon_0^4 < \epsilon_0^2,$$

but this is a contradiction.

2) Now, we suppose (6.5) for $n = 1, 2 \ldots, k$, with $k \geq 2$ a positive integer, then we also will prove (6.5) for $n = k+1$. In fact, in similar way as in case 1), we suppose by contradiction that $\|\partial_x Iu(k+1)\|_{L^2}^2 > 2\epsilon_0^2$, then there exist $t_0 \in (0, k+1)$ such that $\|\partial_x Iu(t_0)\|_{L^2}^2 = 2\epsilon_0^2$. Similarly as in the case 1), from (6.1) we have

$$|E_2(t_0) - 2\epsilon_0^2| \leq 5c\epsilon_0^4, \tag{6.6}$$

by (2.6) and (6.2) we get

$$|E_2(t_0) - E_1(0)| \leq |\Lambda_4(\delta_4)(0)| + \left| \sum_{j=1}^{[t_0]} \int_{j-1}^{j} \Lambda_6(\delta_6)(t)\, dt \right| + \left| \int_{[t_0]}^{t_0} \Lambda_6(\delta_6)(t)\, dt \right|,$$

therefore by (3.6) and (5.2) we easily deduce that

$$\begin{aligned} |E_2(t_0) - \epsilon_0^2| &\leq 5c\epsilon_0^4 + (1 + [t_0])\frac{8c}{N^3}\epsilon_0^6 \\ &\leq 5c\epsilon_0^4 + \lambda^3 T \frac{8c}{N^3}\epsilon_0^6. \end{aligned} \tag{6.7}$$

As in the case $k = 1$, by (6.6) and (6.7) we obtain a contradiction if $\lambda^3 T \sim N^3$, consequently we can to iterate this process $m_{\lambda,T} \sim \lambda^3 T$ times if $T \sim \lambda^{-3} N^3$ and by (6.3) if

$$T \sim N^{(12s-3)/(1+2s)}.$$

Hence u is globally well-posed in H^s for all $s > 1/4$.

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