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Multiplicity and concentration for the nonlinear Schrödinger equation with critical frequency

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Abstract

We consider the nonlinear Schrödinger equation

$$\varepsilon^2 \Delta v - V(x)v + |v|^{p-1}v = 0 \qquad \text{in } \mathbb{R}^N, \tag{E}$$

and the limit problem

$$\Delta u + |u|^{p-1}u = 0 \qquad \text{in } \Omega, \tag{L}$$

with boundary condition $u = 0$ on $\partial\Omega$, where $\Omega = \operatorname{int}\{x \in \mathbb{R}^N : V(x) = \inf V = 0\}$ is assumed to be non-empty, connected and smooth. We prove the existence of an infinite number of solutions for (E) and (L) sharing the topology of their level sets, as seen from the Ljusternik–Schnirelman scheme. Denoting their solutions as $\{v_{k,\varepsilon}\}_{k\in\mathbb{N}}$ and $\{u_k\}_{k\in\mathbb{N}}$, respectively, we show that for fixed $k \in \mathbb{N}$ and, up to rescaling $v_{k,\varepsilon}$, the energy of $v_{k,\varepsilon}$ converges to the energy of u_k. It is also shown that the solutions $v_{k,\varepsilon}$ for (E) concentrate exponentially around Ω and that, up to rescaling and up to a subsequence, they converge to a solution of (L).

1. Introduction

The nonlinear Schrödinger equation, which appears frequently in many fields of physics typically takes the form

$$i\hbar\Psi_t + \frac{\hbar^2}{2}\Delta\Psi - V_o(x)\Psi + |\Psi|^{p-1}\Psi = 0, \qquad \forall x \in \mathbb{R}^N, \forall t \geq 0, \tag{1.1}$$

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with $p > 1$. In this paper we are concerned with the existence and qualitative properties of standing wave solutions of (1.1), that is solutions having the form $\Psi(x,t) = v(x) \cdot \mathrm{e}^{-iEt/\hbar}$. We are especially interested in studying the behavior of the solutions as $\hbar$ approaches zero, the semi-classical limit. In terms of v, the problem can be written as

$$\begin{cases} \varepsilon^2 \Delta v - V(x)v + |v|^{p-1}v = 0, & \text{in } \mathbb{R}^N; \\ v(x) \to 0, & \text{as } |x| \to \infty \end{cases} \qquad (P_\varepsilon)$$

where $\varepsilon^2 = \hbar^2/2$ and $V(x) = V_0(x) - E$. Here we also assume that $N \geq 3$ and $1 + p \in (2, 2^*)$, with $2^* = \frac{2N}{N-2}$.

There has been an enormous amount of research done in the case where the potential V is assumed to be positive. This research was started in the seminal work of Floer and Weinstein [16], where it was shown that in the one dimensional case, for $p = 3$, there is a family of solutions concentrating around a non-degenerate critical point of the potential. These solutions v_ε, which are captured using a Lyapunov–Schmidt reduction method, satisfy

$$\liminf_{\varepsilon \to 0} \max_{x \in \mathbb{R}^N} |v_\varepsilon(x)| > 0. \qquad (1.2)$$

Further research and developments have been carried out by many authors, see e.g. Oh [21], Wang [24], Rabinowitz [23], del Pino and Felmer [11,12], Ambrosetti et al. [1], Gui [17], Li [20], Dancer and Yan [9], Kang and Wei [19] and many others.

In such works, the solutions found satisfy (1.2) and concentrate at certain critical points of the potential, while decaying to zero exponentially, away from them. These works use different approaches, based either on the variational method, or the Lyapunov–Schmidt reduction, or a combination of these. In all these cases, the properties of the positive solutions of the limiting equation are extensively used to obtain the results. In particular, in the Lyapunov–Schmidt reduction approach, the uniqueness and non-degeneracy properties of the positive solution of the limit equation are used.

In contrast with the positive-potential case, there are some recent works by Byeon and Wang [6,7] where they consider a non-negative potential vanishing in a bounded set $\Omega \subset \mathbb{R}^N$. The first important feature is that for the solutions found in [6] and [7], (1.2) no longer holds; actually the maximum value of the solutions approaches zero. The rate at which these solutions vanish depends on the nature of the set Ω. The authors distinguish three cases: (1) $\Omega = \operatorname{int} \overline{\Omega} \neq \emptyset$, which is referred to as the *flat case*, (2) $\overline{\Omega}$ is a finite set of points, and V vanishes polynomially at Ω, which is referred to as the *finite case*, and (3) $\overline{\Omega}$ is a finite set of points, and V vanishes exponentially, which is referred to as the *infinite case*. These rates are certainly strongly related to the nature of the limiting equations.

In this paper we consider the flat case: the interior of the set $\overline{\Omega}$, where the potential V vanishes, is a non-empty bounded set. In this situation the limiting equation is

$$\begin{cases} \Delta u + |u|^{p-1}u = 0, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P)$$

and Byeon and Wang proved in [6] that least energy solutions for (P_ε) converge, up to proper scaling, to a least energy solution for (P). Moreover, they showed that the least energy solutions for (P_ε) concentrate in Ω, by proving their exponential decay outside Ω. In the case where Ω has several connected components, the authors can prescribe in which component the concentration will take place. See also the work of Ding and Tanaka [14].

If we further analyze the limiting problem (P) we realize that besides the least energy solutions, there are many more solutions. In particular, the application of the Ljusternik–Schnirelman theory for even functionals gives the existence of infinitely many solutions. It is quite natural to ask then if problem (P_ε) has infinitely many solutions and what the relation between them and those of (P) is. In this article we answer this question in the case where Ω is connected, we prove that (P_ε) has infinitely many solutions of Ljusternik–Schnirelman type, whose critical levels converge to those of (P). Moreover, we prove that these solutions also concentrate in Ω.

Now we present our results in precise terms. We assume that the potential $V(x)$ verifies:

(V1) V is a continuous non-negative function on $\mathbb{R}^N$;
(V2) $V(x) \to \infty$, as $|x| \to \infty$;
(V3) $\Omega = \mathrm{int}\{x \in \mathbb{R}^N \mid V(x) = 0\} \neq \emptyset$ is connected and smooth.

We consider the functional

$$J_\varepsilon(w) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla w|^2 + \frac{1}{\varepsilon^2}V(x)w^2\right)\mathrm{d}x, \tag{1.3}$$

defined on $\mathcal{M}_\varepsilon = \{w \in H_\varepsilon : \|w\|_{L^{p+1}(\mathbb{R}^N)} = 1\}$, where

$$H_\varepsilon \equiv \left\{w \in \mathrm{H}^1(\mathbb{R}^N) : \|w\|_\varepsilon \equiv \left(\int_{\mathbb{R}^N}|\nabla w|^2 + \frac{V(x)}{\varepsilon^2}w^2\right)^{1/2} < \infty\right\}.$$

The critical points of J_ε on $\mathcal{M}_\varepsilon$ give rise, through scaling, to the solutions of (P_ε).

In our context, the flat case of Byeon and Wang in [6], the limit equation for (P_ε) is (P). Associated to (P) we consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x, \tag{1.4}$$

defined on $\mathcal{M} = \{u \in \mathrm{H}_0^1(\Omega) : \|u\|_{L^{p+1}(\Omega)} = 1\}$. The critical points of J on $\mathcal{M}$ are, up to scaling, the solutions of (P).

Remark 1.1. A family of functions $\{f_\varepsilon\}_{\varepsilon>0}$ is said to sub-converge in a space X, as $\varepsilon \to 0$, when from any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero it is possible to extract a subsequence $\{\varepsilon_{n_i}\}_{i\in\mathbb{N}}$ such that $\{f_{\varepsilon_{n_i}}\}_{i\in\mathbb{N}}$ converge in X, as $i \to \infty$.

Let us state our main result:

Theorem 1.1. *Under our general assumptions on the potential,* (V1), (V2) *and* (V3), *and assuming that* $N \geq 3$ *and* $1 < p < (N+2)/(N-2)$ *we have:*

(i) *Given* $\varepsilon > 0$ *the functional* J_ε *possesses infinitely many critical points* $\{\hat{w}_{k,\varepsilon}\}_{k\in\mathbb{N}} \subset \mathcal{M}_\varepsilon$.
(ii) *The limit functional* J *possesses infinitely many critical points* $\{\hat{w}_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$.
(iii) *Given* $k \in \mathbb{N}$, *the critical values satisfy*

$$\lim_{\varepsilon\to 0} J_\varepsilon(\hat{w}_{k,\varepsilon}) = J(\hat{w}_k). \tag{1.5}$$

(iv) *Moreover, given* $\delta, c > 0$, *there exists* $\varepsilon_0 > 0$ *such that*

$$|\hat{w}_{k,\varepsilon}(x)| < C \cdot \exp\left\{-\frac{c}{\varepsilon}\cdot \mathrm{dist}(x, \Omega^\delta)\right\}, \qquad \forall x \in \mathbb{R}^N,\ \forall \varepsilon \in]0, \varepsilon_0), \tag{1.6}$$

where $C > 0$ *and* $\Omega^\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < \delta\}$.

(v) *On the boundary of* Ω, *the functions* $\hat{w}_{k,\varepsilon}$ *verify*

$$\lim_{\varepsilon\to 0}\max_{x\in\partial\Omega}|\hat{w}_{k,\varepsilon}(x)| = 0, \qquad \forall k\in\mathbb{N}. \tag{1.7}$$

It is clear that the functions

$$v_{k,\varepsilon} = (2\varepsilon^2 c_{k,\varepsilon})^{1/(p-1)}\hat{w}_{k,\varepsilon}, \qquad c_{k,\varepsilon} = J_\varepsilon(\hat{w}_{k,\varepsilon}),$$

are solutions of (P_ε) and, as a corollary, they satisfy, for fixed $k\in\mathbb{N}$,

$$\lim_{\varepsilon\to 0}\|v_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} = 0 \tag{1.8}$$

and

$$\liminf_{\varepsilon\to 0}\frac{\|v_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)}}{\varepsilon^{2/(p-1)}} > 0. \tag{1.9}$$

It is not hard to see that the functions $w_{k,\varepsilon} = (2c_{k,\varepsilon})^{1/(p-1)}\hat{w}_{k,\varepsilon}$ satisfy the equation

$$\begin{cases}\Delta w - \varepsilon^{-2}V(x)w + |w|^{p-1}w = 0, & \text{in } \mathbb{R}^N;\\ w(x)\to 0, & \text{as } |x|\to\infty.\end{cases} \tag{P'_ε}$$

We prove, for every k, the existence of a subsequence of $w_{k,\varepsilon}$ converging to w_k, a solution of (P).

The situation described for our sequence of solutions corresponds to the same phenomena as were discussed by Byeon and Wang, ([6, Theorem 2.2]), for (positive) least energy solutions. Property (1.8) is in contrast to the non-critical case, $\inf_{x\in\mathbb{R}^N}V(x) > 0$, where all the solutions of (P_ε) are bounded away from zero.

In [6] it is shown that the rescaled function $w_\varepsilon = \varepsilon^{-2/(p-1)}v_\varepsilon$ sub-converges pointwise to a least energy solution U of (P), in Ω, and to 0 in $\mathbb{R}^N\setminus\Omega$. Moreover, given $\delta > 0$, the convergence is uniform on $\{x\in\mathbb{R}^N : \mathrm{dist}(x,\partial\Omega)\ge\delta\}$.

Remark 1.2. The potential considered in this article is non-negative and vanishing at an open set Ω. This situation is considered critical since the limiting behavior of solutions is quite different. For positive potential least energy solutions must concentrate at a point, however for a vanishing potential concentration occurs at the whole set Ω. When the potential becomes negative in a bounded set, then least energy solutions no longer make sense. However, this situation can still be well understood, at least in the one dimensional case, and in the radial case, as in the work by Felmer and Torres [15] and Castro and Felmer [8], respectively.

Remark 1.3. We do not say anything about the sign of the solutions we found; however, since the limit problem (P) may have many positive solutions depending on the geometry of Ω (see e.g. [10]), the same could happen with (P_ε).

Remark 1.4. In this article we consider only the case of a potential diverging to infinity as $|x|\to\infty$, that is satisfying (V2), and vanishing in a connected, open, smooth set, that is satisfying (V3). We think that our results hold for more general potentials, when the zero set of V is not connected, and also for the finite and infinite cases. Particularly challenging may be the case of a bounded potential, positive at infinity. In this case the existence of infinitely many critical points as in Theorem 1.1(i) may be no longer true. However, the statements, (ii)–(v), with k fixed, should be true.

Actually, after we finished this article, we learned of a recent work of Ding and Szulkin [13] where this problem is treated. Instead of conditions (V2) and (V3) they assume that

(V2′) there exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) < b\}$ is nonempty and has finite measure.

Fixed $k \in \mathbb{N}$, they prove that for some $\Lambda_k > 0$, problem (P_ε) has at least k pairs of solutions in H_ε when $\varepsilon \in]0, \Lambda_k^{-1/2}[$. In contrast we prove the existence of an infinite number of solutions, at least a pair for each level of energy. If, for every $m \in \mathbb{N}$, u_m is a solution of (P_{ε_m}), where $\lim_{m\to\infty} \varepsilon_m = 0$, then they show that u_m converges in H_1 to some solution u of (P) assuming the boundedness of $(\|u_m\|_{\varepsilon_m})_{m\in\mathbb{N}}$. In Lemma 3.1 we prove this last condition for each level k of energy and, in Lemma 4.2, we prove that our solutions $(w_{k,\varepsilon})$ subconverge in $\mathrm{H}^1(\mathbb{R}^N)$ to some solution of (P).

We observe that as far as the existence and the number of solutions are concerned, the problem

$$\begin{cases} \Delta v - V_\lambda(x)v + |v|^{p-1}v = 0, & \text{in } \mathbb{R}^N; \\ v(x) \to 0, & \text{as } |x| \to \infty, \end{cases} \tag{P_λ}$$

where $V_\lambda = \lambda V$, is equivalent to (P_ε). In fact, putting $\varepsilon^2 = \lambda^{-1}$, it is clear that u is a solution of (P_λ) if and only if $v = \lambda^{-1/(p-2)}u$ is a solution of (P_ε). In some recent work, Bartsch and Wang [3] and Bartsch et al. [2] dealt with problem (P_λ) when $V_\lambda(x) = a_0(x) + \lambda a(x)$, where $a_0 \in \mathrm{L}^\infty(\mathbb{R}^N)$ is bounded away from zero, and $a \in \mathrm{L}^\infty(\mathbb{R}^N)$ is non-negative and such that for some $M_0 > 0$ and some $Z = \overline{Z} \subset \mathbb{R}^N$ with non-empty interior,

$$a(x) = 0, \qquad \forall x \in Z \qquad \text{and} \qquad a(x) > 0, \qquad \text{a.e. } x \in Z^c,$$

and

$$|\{x \in \mathbb{R}^N : a(x) < M_0\}| < \infty.$$

They show that for every integer $k \in \mathbb{N}$, there exists Λ_k such that (P_λ) has at least k pairs of (weak) solutions when $\lambda > \Lambda_k$; with additional conditions these solutions have exponential decay at infinity. They prove that a sequence $\{u_n\}_{n\in\mathbb{N}}$ of solutions for (P_{λ_n}), $\lambda_n \longrightarrow \infty$, converge in $\mathrm{H}^1(\mathbb{R}^N)$ to a solution of

$$\begin{cases} -\Delta u + a_0(x)u = |u|^{p-1}, & \text{in int } Z, \\ u = 0, & \text{in } Z^c, \end{cases}$$

provided there is uniform boundedness of the energy norms of $\{u_n\}_{n\in\mathbb{N}}$ and $\inf \|u_n\|_{L^p(\mathbb{R}^N)} > 0$.

We finally mention that in our work we not only obtain exponential decay of the solutions at infinity, but we also get further asymptotic estimates on their behavior on the boundary of the domain, see Section 5.

We devote the paper to proving Theorem 1.1. In Section 2 we set up the Ljusternik–Schnirelman scheme to prove parts (i) and (ii) of Theorem 1.1. In Section 3, we study the asymptotic behavior of the critical values proving (iii) of Theorem 1.1. In Section 4 we analyze the decay of the solutions away from Ω and in Section 5 we study the behavior on the boundary, proving (iv) and (v), respectively.

2. Ljusternik–Schnirelman setting: Multiplicity

In this section we set up the Ljusternik–Schnirelman scheme in order to prove the first two statements in Theorem 1.1. In general terms, given a Banach space E, we write

$\Sigma_E = \{A \subset E : A = \overline{A}, A = -A, 0 \notin A\}$ and consider in Σ_E Krasnoselski's genus γ (see e.g. Rabinowitz [22]). The following theorem is proved in [22].

Theorem 2.1. *Let $M \in \Sigma_E$ be C^1 sub-manifold of E and let $f \in \mathrm{C}^1(E)$ be even. Suppose that (M,f) satisfy the Palais–Smale condition and let*

$$C_k(f) = \inf_{A \in \mathcal{A}_k(M)} \max_{u \in A} f(u), \tag{2.1}$$

where

$$\mathcal{A}_k(M) = \{A \in \Sigma_E \cap M : \gamma(A) \geq k\}. \tag{2.2}$$

If $C_k(f) \in \mathbb{R}$, then $C_k(f)$ is a critical value for f. Moreover, if $c \equiv C_k(f) = \cdots = C_{k+m}(f)$, then $\gamma(K_c) \geq m + 1$. In particular, if $m > 1$, then K_c, the set of critical points corresponding to the value c, contains infinitely many elements.

It is clear that the functional (1.4) verifies the conditions of Theorem 2.1. Then we write $\Sigma = \Sigma_{\mathrm{H}_0^1(\Omega)}$, and for each $k \in \mathbb{N}$,

$$\mathcal{A}_k = \mathcal{A}_k(\mathcal{M}) \qquad \text{and} \qquad c_k = C_k(J) = J(\hat{w}_k) \in (0, \infty),$$

Remark 2.1. With this it is clear that

$$w_k \equiv (2c_k)^{1/(p-1)} \cdot \hat{w}_k$$

is a solution of (P).

In our study it will be convenient to have an intermediate problem. Given $\delta > 0$ we write $\Omega^\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, \Omega) < \delta\}$, and consider the problem

$$\begin{cases} \Delta u + |u|^{p-1}u = 0, & \text{in } \Omega^\delta, \\ u = 0, & \text{on } \partial\Omega^\delta \end{cases} \tag{P^δ}$$

with the functional

$$J^\delta(u) \equiv \frac{1}{2}\int_{\Omega^\delta} |\nabla u|^2 \mathrm{d}x \tag{2.3}$$

defined on $\mathcal{M}^\delta = \{u \in \mathrm{H}_0^1(\Omega^\delta) : \|u\|_{L^{p+1}(\Omega^\delta)} = 1\}$. Here we write $\Sigma^\delta = \Sigma_{\mathrm{H}_0^1(\Omega^\delta)}$, and for each $k \in \mathbb{N}$

$$\mathcal{A}_k^\delta = \mathcal{A}_k(\mathcal{M}^\delta) \qquad \text{and} \qquad c_k^\delta = C_k(J^\delta) = J^\delta(\hat{w}_k^\delta) \in (0, +\infty).$$

It is clear that the function $w_k^\delta = (2c_k^\delta)^{1/(p-1)}\hat{w}_k^\delta$ is a solution of (P^δ).

Theorem 2.1 can also be applied to (P_ε). In fact, the compactness of the embedding $H_\varepsilon \subset \mathrm{L}^q(\mathbb{R}^N)$, $q \in [2, 2^*)$, can be proved applying the Fréchet–Kolmogorov theorem ([4, Corollary IV.26]); and with this, it is proved that the corresponding functional is C^1 and satisfies the Palais–Smale condition in the manifold $\mathcal{M}_\varepsilon$. We put

$$\Sigma_\varepsilon = \Sigma_{H_\varepsilon}, \qquad \forall \varepsilon > 0,$$

and, for every $k \in \mathbb{N}$ and every $\varepsilon > 0$,

$$\mathcal{A}_{k,\varepsilon} = \mathcal{A}_k(\mathcal{M}_\varepsilon) \qquad \text{and} \qquad c_{k,\varepsilon} = C_k(J_\varepsilon) = J_\varepsilon(\hat{w}_{k,\varepsilon}).$$

Remark 2.2. With this it is clear that

$$v_{k,\varepsilon} = (2\varepsilon^2 c_{k,\varepsilon})^{1/(p-1)} \cdot \hat{w}_{k,\varepsilon}$$

is a solution of (P_ε) and

$$w_{k,\varepsilon} \equiv (2c_{k,\varepsilon})^{1/(p-1)} \cdot \hat{w}_{k,\varepsilon}$$

is a solution of (P'_ε).

Remark 2.3. Assuming further that the potential is of class C^α, using the well-known regularity theory, it can be proved that each 'solution' which appears in this paper is a classical one and belongs to the class $\mathrm{C}^{2,\alpha}$.

3. Limits for the critical values

This section is devoted to proving (iii) of Theorem 1.1. As discussed in the last section, the multiplicity result is based on the Ljusternik–Schnirelman theory for even functionals. The indices k of the critical values represent the topological characteristic of the level set, as captured by the Krasnoselski genus.

Thus, our main result in this section corresponds to proving that the level sets of J_ε and J for the Ljusternik–Schnirelman values are topologically equivalent. Actually we prove

Theorem 3.1. *For every $k \in \mathbb{N}$, we have*

$$\lim_{\varepsilon\to 0} c_{k,\varepsilon} = c_k. \tag{3.1}$$

The proof of this theorem is divided into several steps as given by the following lemmas.

Lemma 3.1. *For every $k \in \mathbb{N}$ and every $\varepsilon > 0$, we have*

$$c_{k,\varepsilon} \le c_k. \tag{3.2}$$

Proof. If we identify each $u \in \mathrm{H}_0^1(\Omega)$ with its extension by zero outside Ω then we have $\mathrm{H}_0^1(\Omega) \subset H_\varepsilon$. We also have that $\|u\|_\varepsilon = \|u\|_{H_0^1(\Omega)}$, for all $u \in \mathrm{H}_0^1(\Omega)$, and clearly $\mathcal{A}_k \subset \mathcal{A}_{k,\varepsilon}$. Hence $c_{k,\varepsilon} \le c_k$, for every $k \in \mathbb{N}$. □

Now the crucial lemma

Lemma 3.2. *Let $k \in \mathbb{N}$ and $\sigma > 0$. Given $\delta > 0$ small, there exists a $\varepsilon_\delta > 0$ such that*

$$c_k^\delta \le c_{k,\varepsilon} + \sigma, \tag{3.3}$$

for every $\varepsilon \in (0, \varepsilon_\delta)$.

Proof. According to the definition of the $c_{k,\varepsilon}$, given $\varepsilon > 0$ (in principle without restrictions), we choose $A_\sigma(\varepsilon) \in \mathcal{A}_{k,\varepsilon}$ in such a way that

$$\max_{v\in A_\sigma(\varepsilon)} J_\varepsilon(v) \le c_{k,\varepsilon} + \frac{\sigma}{3} \tag{3.4}$$

holds. Then, by Lemma 3.1,

$$J_\varepsilon(v) \le c_k + \frac{\sigma}{3} \equiv b_{k,\sigma}, \qquad \forall v \in A_\sigma(\varepsilon). \tag{3.5}$$

From here we directly obtain that

$$\frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 \le b_{k,\sigma}, \qquad \forall v \in A_\sigma(\varepsilon) \tag{3.6}$$

and

$$b_{k,\sigma} \ge \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega}\frac{V(x)}{\varepsilon^2}v^2 \ge \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega^\delta}\frac{V(x)}{\varepsilon^2}v^2, \qquad \forall v \in A_\sigma(\varepsilon) \tag{3.7}$$

$\forall\delta > 0$. Here we notice that the constant $b_{k,\sigma}$ does not depend on ε. Now, putting $V_\rho \equiv \inf_{x\in\mathbb{R}^N\setminus\Omega^\rho} V(x)$, we have

$$\|v\|_{L^2(\mathbb{R}^N\setminus\Omega^\delta)} \le \left(\frac{2b_{k,\sigma}}{V_\delta}\right)^{1/2}\varepsilon, \qquad \forall\delta > 0,\ \forall v \in A_\sigma(\varepsilon). \tag{3.8}$$

From (3.6) and using the Sobolev–Gagliardo–Nirenberg inequality we get

$$\|v\|_{L^{2^*}(\mathbb{R}^N)} \le C b_{k,\sigma}^{1/2}, \qquad \forall v \in A_\sigma(\varepsilon), \tag{3.9}$$

for some constant C. Thus, we conclude

$$\lim_{\varepsilon\to 0}\max_{v\in A_\sigma(\varepsilon)}\|v\|_{L^{p+1}(\mathbb{R}^N\setminus\Omega^\delta)} = 0, \qquad \forall\delta > 0. \tag{3.10}$$

In fact, getting $\alpha \in (0,1)$ such that $\frac{1}{p+1} = \frac{(1-\alpha)}{2} + \frac{\alpha}{2^*}$, it follows by interpolation, considering (3.8) and (3.9), that

$$\|v\|_{L^{p+1}(\mathbb{R}^N\setminus\Omega^\delta)} \le \eta\varepsilon^{1-\alpha}, \qquad \forall v \in A_\sigma(\varepsilon),\ \forall\delta > 0, \tag{3.11}$$

with

$$\eta = \eta(\delta,k,\sigma,q) = C\left(\frac{b_{k,\sigma}}{V_\delta^{1-\alpha}}\right)^{1/2}.$$

From (3.10) it is clear that, given $\delta > 0$ and $s > 0$, we can get a $\varepsilon_1 = \varepsilon_1(\delta,s) > 0$ such that

$$\max_{v\in A_\sigma(\varepsilon)}\|v\|_{L^{p+1}(\mathbb{R}^N\setminus\Omega^\delta)} \le \delta^s, \qquad \forall\varepsilon \in (0,\varepsilon_1[\tag{3.12}$$

and thus, in particular for $s = 1$,

$$\|v\|_{L^{p+1}(\Omega^\delta)} \ge 1-\delta, \qquad \forall v \in A_\sigma(\varepsilon),\ \forall\varepsilon \in (0,\varepsilon_1[, \forall\delta > 0. \tag{3.13}$$

From now on we will assume that $0 < \delta < 1$. We choose a cut-off function $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_\delta \equiv 1$ in $\Omega^{\delta/2}$ and $\phi_\delta \equiv 0$ in $\mathbb{R}^N \setminus \Omega^\delta$,

$$0 < \phi_\delta(x) < 1 \quad\text{and}\quad |\nabla\phi_\delta(x)| \le \frac{1}{\delta^r} \qquad \forall x \in \Omega^\delta \setminus \overline{\Omega^{\delta/2}}, \tag{3.14}$$

for some $r > 1$. Now we define for $u \in \mathcal{M}_\varepsilon$

$$\phi_\delta[u] \equiv \frac{\phi_\delta u}{\|\phi_\delta u\|_{L^{p+1}(\mathbb{R}^N)}}, \tag{3.15}$$

and we claim that

$$\phi_\delta[A_\sigma(\varepsilon)] \in \mathcal{A}_k^\delta, \qquad \forall\varepsilon \in (0,\varepsilon_1). \tag{3.16}$$

In fact, as a consequence of the concentration property given in (3.13), for all $v \in A_\sigma(\varepsilon)$, for all $\varepsilon \in (0, \varepsilon_1)$,

$$\int_{\Omega^\delta} |\phi_\delta v|^{p+1} = \int_{\Omega^{\delta/2}} |v|^{p+1} + \int_{\Omega^\delta \setminus \Omega^{\delta/2}} |\phi_\delta v|^{p+1} \geq \left(1 - \frac{\delta}{2}\right)^{p+1}, \ \forall \delta \in]0,1[,$$

so that

$$\|\phi_\delta v\|_{L^{p+1}(\Omega^\delta)} \geq 1 - \delta, \tag{3.17}$$

and in particular we see that $\phi_\delta[\cdot]$ is well defined and we further conclude that it is continuous. Then, since $\phi_\delta[\cdot]$ is odd, from genus properties we have that

$$\gamma(\phi_\delta[A_\sigma(\varepsilon)]) \geq k, \qquad \forall \varepsilon \in (0, \varepsilon_1).$$

Hence, considering (3.16) and the definition of c_k^δ, we get

$$c_k^\delta \leq \max_{v \in \phi_\delta[A_\sigma(\varepsilon)]} J^\delta(v), \qquad \forall \varepsilon \in (0, \varepsilon_1). \tag{3.18}$$

Let us take now an element $u \in A_\sigma(\varepsilon)$ such that $\bar{v} \equiv \phi_\sigma[u]$ satisfies

$$\max_{v \in \phi_\delta[A_\sigma(\varepsilon)]} J^\delta(v) \leq J^\delta(\bar{v}) + \frac{1}{3}\sigma. \tag{3.19}$$

At this stage, we observe that in order to complete the proof of the lemma it is enough to prove the existence of an element $w \in A_\sigma(\varepsilon)$ satisfying

$$J^\delta(\bar{v}) \leq J_\varepsilon(w) + \frac{1}{3}\sigma. \tag{3.20}$$

In fact, from (3.4) and (3.18)–(3.20), we have

$$c_k^\delta \leq J^\delta(\bar{v}) + \frac{1}{3}\sigma \leq J_\varepsilon(w) + \frac{2}{3}\sigma \leq \max_{u \in A_\sigma(\varepsilon)} J_\varepsilon(u) + \frac{2}{3}\sigma \leq c_{k,\varepsilon} + \sigma.$$

We devote the rest of the proof to finding such a w. For $\bar{v} = \phi_\sigma[u]$, a direct computation gives

$$\begin{aligned} \|\phi_\delta u\|^2_{L^{p+1}(\mathbb{R})} J^\delta(\bar{v}) &\leq \int_{\Omega^\delta} u^2|\nabla\phi_\delta|^2 + 2u\phi_\delta \nabla u \nabla \phi_\delta + \phi_\delta^2 |\nabla u|^2 \\ &\leq \int_{\Omega^\delta} u^2|\nabla\phi_\delta|^2 + 2u\phi_\delta \nabla u \nabla \phi_\delta + \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{V(x)}{\varepsilon^2} u^2 \end{aligned}$$

whence

$$\begin{aligned} (1-\delta)^2 J^\delta(\bar{v}) &\leq J_\varepsilon(u) + \int_{\Omega^\delta \setminus \overline{\Omega^{\delta/2}}} u^2|\nabla\phi_\delta|^2 + 2u|\nabla u||\nabla\phi_\delta| \\ &\leq J_\varepsilon(u) + \frac{1}{\delta^{2r}} \int_{\Omega^\delta \setminus \overline{\Omega^{\delta/2}}} u^2 + \frac{2}{\delta^r} \int_{\Omega^\delta \setminus \overline{\Omega^{\delta/2}}} u|\nabla u| \\ &\leq J_\varepsilon(u) + \frac{C}{\delta^{2r}} \left(\int_{\Omega^\delta \setminus \overline{\Omega^{\delta/2}}} u^2 \right)^{1/2} \end{aligned}$$

where we have used (3.14), (3.17), (1.3), (3.6) and the Cauchy–Schwartz inequality. We observe that the constant C depends on k through $b_{k,\sigma}$. Then, using Hölder inequality, considering (3.12) and taking $s > 2r$, we get decreasing ε_1 if necessary

$$(1-\delta)^2 J^\delta(\bar{v}) \leq J_\varepsilon(u) + C\delta^{s-2r}, \qquad \forall \varepsilon \in (0, \varepsilon_1). \tag{3.21}$$

Here, if $\delta \in \left(0, \frac{1}{4}\right)$ then $\frac{1}{2} J^{\delta}(\bar{v}) \leq J_{\varepsilon}(u) + C\delta^{s-2r} \leq b_{k,\sigma} + C\delta^{s-2r}$. So, from (3.21) we get

$$J^{\delta}(\bar{v}) \leq J_{\varepsilon}(u) + C\delta^{s-2r} + 2\delta(b_{k,\sigma} + C\delta^{s-2r}).$$

From here we obtain (3.20) putting $u = w$ when $\delta \in \left(0, \frac{1}{4}\right)$ is small enough and $\varepsilon \in (0, \varepsilon_1)$. □

Lemma 3.3. *Given $k \in \mathbb{N}$ and $\delta > 0$, we have*

$$c_k^{\delta} \leq c_k.$$

Proof. We identify each $u \in \mathrm{H}_0^1(\Omega)$ with its extension by zero to $\Omega^{\delta} \setminus \overline{\Omega}$. In this sense we have $\mathrm{H}_0^1(\Omega) \subset \mathrm{H}_0^1(\Omega^{\delta})$ and $\|u\|_{H_0^1(\Omega^{\delta})} = \|u\|_{H_0^1(\Omega)}$, for all $u \in \mathrm{H}_0^1(\Omega)$. Thus, it is clear that $\mathcal{A}_k \subset \mathcal{A}_k^{\delta}$ and then $c_k^{\delta} \leq c_k$, for every $k \in \mathbb{N}$. □

Lemma 3.4. *Given $k \in \mathbb{N}$ and $\sigma > 0$, there exists $\delta_{\sigma} > 0$ such that*

$$c_k \leq c_k^{\delta} + \sigma,$$

for every $\delta \in (0, \delta_{\sigma})$.

Proof. According to the definition of c_k^{δ}, given $\delta > 0$ we may choose $B_{\sigma}(\delta) \in \mathcal{A}_k^{\delta}$ such that

$$\max_{v \in B_{\sigma}(\delta)} J^{\delta}(v) \leq c_k^{\delta} + \frac{\sigma}{3}. \tag{3.22}$$

Then, from Lemma 3.3, we get

$$J^{\delta}(v) \leq c_k + \frac{\sigma}{3} \equiv b_{k,\sigma}, \qquad \forall v \in B_{\sigma}(\delta). \tag{3.23}$$

Now we choose a $\delta_0 = \delta_0(\Omega) > 0$ so that for every $\delta \in (0, \delta_0)$ we can associate a diffeomorphism $\psi_{\delta} = (\psi_{\delta}^{(1)}, \ldots, \psi_{\delta}^{(N)}) \in \mathrm{C}^1(\overline{\Omega}; \overline{\Omega^{\delta}})$ such that

$$|\psi_{\delta}(x) - x| \leq O(\delta) \qquad \text{and} \qquad |D\psi_{\delta}(x) - I_N| \leq O(\delta) \qquad \forall x \in \overline{\Omega}, \tag{3.24}$$

and

$$\psi_{\delta}(\partial\Omega) = \partial\Omega^{\delta}. \tag{3.25}$$

Here I_N denotes the $N \times N$ identity matrix. Now we define the application $\Gamma_{\delta}[\cdot] : \mathrm{H}_0^1(\Omega^{\delta}) \to \mathrm{H}_0^1(\Omega)$ as

$$\Gamma_{\delta}[v](x) = \frac{v \circ \psi_{\delta}(x)}{\|v \circ \psi_{\delta}\|_{L^{p+1}(\Omega)}}, \qquad x \in \Omega,$$

for all $v \in \mathrm{H}_0^1(\Omega^{\delta}) \setminus \{0\}$. We claim that there exists $\delta_1 \in (0, \delta_0)$ such that

$$\Gamma_{\delta}[B_{\sigma}(\delta)] \in \mathcal{A}_k, \qquad \forall \delta \in (0, \delta_1). \tag{3.26}$$

We see that in order to prove (3.26) it suffices to show that Γ_{δ} is well defined and continuous, since clearly Γ_{δ} is odd. We do this now.

First, we observe that from (3.24), for every $\eta > 0$, there exists a $\delta_2 = \delta_2(\eta) > 0$, $\delta_2 \leq \delta_1$, such that

$$1 - \eta \leq \det D\psi_{\delta}(x) \leq 1 + \eta, \qquad \forall x \in \overline{\Omega}, \ \forall \delta \in (0, \delta_2). \tag{3.27}$$

From now on we assume that $\delta \in (0, \delta_2)$. Let v be an arbitrary element in $\mathrm{H}_0^1(\Omega^\delta) \setminus \{0\}$. Then, from (3.27) and the formula of change of variables we get

$$\begin{aligned}\|v \circ \psi_\delta\|_{L^{p+1}(\Omega)}^{p+1} &\geq |1+\eta|^{-1} \int_\Omega |v \circ \psi_\delta|^{p+1} \det D\psi_\delta(x) \\ &\geq |1+\eta|^{-1} \int_{\Omega^\delta} |v|^{p+1}.\end{aligned}$$

Thus, in particular $v \circ \psi_\delta \neq 0$, for every $v \in \mathrm{H}_0^1(\Omega^\delta)$. Using again (3.27) we obtain that for all $v \in \mathrm{H}_0^1(\Omega^\delta) \setminus \{0\}$

$$\frac{\|v\|_{L^{p+1}(\Omega^\delta)}}{|1+\eta|^{1/(p+1)}} \leq \|v \circ \psi_\delta\|_{L^{p+1}(\Omega)} \leq \frac{\|v\|_{L^{p+1}(\Omega^\delta)}}{|1-\eta|^{1/(p+1)}}.$$

Let us prove next that

$$\Gamma_\delta(v) \in \mathrm{H}_0^1(\Omega) \setminus \{0\}, \qquad \forall v \in \mathrm{H}_0^1(\Omega^\delta) \setminus \{0\}. \tag{3.28}$$

Let $i \in \{1, \ldots, N\}$ and $w \in \mathrm{C}_0^\infty(\Omega^\delta) \setminus \{0\}$, then we have

$$D_i \Gamma_\delta[w](x) = \frac{\sum_{j=1}^N g_{i,j}(x)}{\|w \circ \psi_\delta\|_{L^{p+1}(\Omega)}}, \qquad x \in \overline{\Omega},$$

where $g_{i,j}(x) = D_j w(\psi_\delta(x)) \cdot D_i \psi_\delta^{(j)}(x)$. Then, from (3.27) and using the formula of change of variables again, we get

$$\begin{aligned}\int_\Omega \left| \sum_{j=1}^N g_{i,j}(x) \right|^2 &\leq \int_{\psi_\delta^{-1}(\Omega^\delta)} \sum_{j=1}^N |g_{i,j}(x)|^2 \frac{\det D\psi_\delta(x)}{\det D\psi_\delta(x)} \\ &\leq C \int_{\Omega^\delta} \sum_{j=1}^N |D_j w|^2,\end{aligned}$$

where $C = (1-\eta)^{-1} \left(\max_{\substack{x\in\Omega \\ j=1,\ldots,N}} |D_i \psi_\delta^{(j)}(x)|^2 \right)$. Moreover, from (3.25) we have $\Gamma_\delta[w]|_{\partial\Omega} = 0$ and then $\Gamma_\delta[w] \in \mathrm{H}_0^1(\Omega)$. Thus we have proved that $\Gamma_\delta[w] \in \mathrm{H}_0^1(\Omega) \setminus \{0\}$ and

$$\|\Gamma_\delta(w)\|_{H_0^1(\Omega)} \leq K \|w\|_{H_0^1(\Omega^\delta)}, \qquad \forall w \in \mathrm{C}_0^\infty(\Omega^\delta), \tag{3.29}$$

for certain $K = K(\eta, \delta, N)$. Using a density argument we extend this inequality to all $\mathrm{H}_0^1(\Omega^\delta)$. From here we obtain (3.28) and the continuity of $\Gamma_\delta[\cdot]$. Finally, from (3.28) and (3.29), we obtain (3.26) proving the claim.

Now, considering (3.26) and the definition of c_k, it follows that

$$c_k \leq \max_{u \in \Gamma_\delta[B_\sigma(\delta)]} J(u). \tag{3.30}$$

On the other hand, let us take $v \in B_\sigma(\delta)$ such that $u = \Gamma_\delta[v]$ satisfies

$$\max_{u^* \in \Gamma_\delta[B_\sigma(\delta)]} J(u^*) \leq J(u) + \frac{\sigma}{3}. \tag{3.31}$$

At this stage, if we find an element $w \in B_\sigma(\delta)$ such that $J(u) \le J^\delta(w) + \frac{\sigma}{3}$ then we complete the proof of the lemma. In fact, from (3.22), (3.30) and (3.31),

$$\begin{aligned} c_k &\le J(u) + \frac{\sigma}{3} \le J^\delta(w) + \frac{2\sigma}{3} \\ &\le \max_{w\in B_\sigma(\delta)} J^\delta(w) + \frac{2\sigma}{3} \le c_k^\delta + \sigma. \end{aligned} \tag{3.32}$$

To finish then, let us find such a w. Choosing $\delta \in (0, \delta_2)$ small enough, for $u = \Gamma_\delta[v]$ we have

$$\begin{aligned} J(u) &= \frac{1}{2}\|v \circ \psi_\delta\|^{-1}_{L^{p+1}(\Omega)} \int_\Omega \sum_{i=1}^N \sum_{j=1}^N |g_{i,j}|^2 \\ &\le \frac{1}{2}(1+\eta)^{1/(p+1)} \int_\Omega \sum_{i=1}^N \sum_{j=1}^N [\delta_{i,j} + O(\delta)]^2 |D_j v(\psi_\delta(x))|^2 \\ &\le (1+\eta)^{1/(p+1)}[1+O(\delta)]^2 \int_\Omega \sum_{j=1}^N |D_j v(\psi_\delta(x))|^2 \frac{\det D\psi_\delta(x)}{(1-\eta)} \\ &\le (1+\eta)^{1/(p+1)} \frac{[1+O(\delta)]^2}{(1-\eta)} J^\delta(v). \end{aligned} \tag{3.33}$$

We see that we can choose $w = v$. Here we used (3.23) and (3.24) and the fact that $\|v\|_{L^{p+1}(\Omega^\delta)} = 1$. □

Proof of Theorem 3.1. Let $\sigma > 0$ be small. Considering Lemma 3.4, we choose a $\delta \in (0, \delta_{\sigma/2})$; then, from Lemma 3.2, there exists a $\varepsilon_\delta > 0$ (implicitly depending on σ) such that $c_k \le c_k^\delta + \sigma/2 \le c_{k,\varepsilon} + \sigma$, for every $\varepsilon \in (0, \varepsilon_\delta)$. Because of Lemma 3.1, we conclude since $\sigma > 0$ is arbitrary. □

4. Asymptotic profiles and concentration phenomena

In this section we study the asymptotic behavior of the solutions, both inside Ω and outside Ω. Throughout this section we use the notation introduced in Section 2.

Lemma 4.1. *For every $k \in \mathbb{N}$, as $\varepsilon \to 0$, $w_{k,\varepsilon}$ sub-converges weakly to a $u_k \in \mathrm{H}^1(\mathbb{R}^N)$ such that its restriction to Ω is a solution of (P), with $J(\hat{u}_k|_\Omega) = c_k$, for $\hat{u}_k = (2c_k)^{1/(1-p)} u_k$.*

Proof. First, we prove that for ε_δ small we have that

$$\|\hat{w}_{k,\varepsilon}\|_{H^1(\mathbb{R}^N)} \le K_1, \qquad \forall \varepsilon \in (0, \varepsilon_\delta), \tag{4.1}$$

where $K_1 > 0$ is a constant, depending only on k. From Lemma 3.1, we get

$$\|\nabla \hat{w}_{k,\varepsilon}\|^2_{L^2(\mathbb{R}^N)} \le 2c_k, \qquad \forall \varepsilon \in (0, \varepsilon_\delta), \tag{4.2}$$

that is

$$\int_{\mathbb{R}^N} |\nabla \hat{w}_{k,\varepsilon}|^2 \le \int_{\mathbb{R}^N} |\nabla \hat{w}_k|^2, \tag{4.3}$$

and then, as a consequence of Gagliardo–Nirenberg inequality,

$$\|\hat{w}_{k,\varepsilon}\|_{L^{2^*}(\mathbb{R}^N)} \le C c_k^{1/2}, \tag{4.4}$$

for some constant C only depending on N. Given $R \geq 1$, we have

$$\begin{aligned}\|\hat{w}_{k,\varepsilon}\|^2_{L^2(\mathbb{R}^N)} &= \|\hat{w}_{k,\varepsilon}\|^2_{L^2(\mathbb{R}^N\setminus\Omega^R)} + \|\hat{w}_{k,\varepsilon}\|^2_{L^2(\Omega^R)} \\ &\leq \frac{2c_k}{V_R}\varepsilon^2 + \|\hat{w}_{k,\varepsilon}\|^2_{L^{2^*}(\Omega^R)} \cdot |\Omega^R|^{2/N} \\ &\leq \frac{2c_k}{V_R} + \|\hat{w}_{k,\varepsilon}\|^2_{L^{2^*}(\mathbb{R}^N)} \cdot |\Omega^R|^{2/N}, \end{aligned} \tag{4.5}$$

where we have used the Hölder inequality and the relation

$$\|\hat{w}_{k,\varepsilon}\|_{L^2(\mathbb{R}^N\setminus\Omega^\delta)} \leq \left(\frac{2c_k}{V_\delta}\right)^{1/2} \varepsilon, \qquad \forall\varepsilon \in (0,\varepsilon_\delta), \forall\delta > 0, \tag{4.6}$$

which comes from (3.8). Then, because of (4.2), (4.4) and putting $R = 1$,

$$\|\hat{w}_{k,\varepsilon}\|^2_{H^1(\mathbb{R}^N)} = \|\nabla\hat{w}_{k,\varepsilon}\|^2_{L^2(\mathbb{R}^N)} + \|\hat{w}_{k,\varepsilon}\|^2_{L^2(\mathbb{R}^N)} \leq K_1.$$

From the estimate (4.1), there exists a $\hat{u}_k \in \mathrm{H}^1(\mathbb{R}^N)$ such that $\hat{w}_{k,\varepsilon}$ sub-converge weakly and pointwise to $\hat{u}_k \in \mathrm{H}^1(\mathbb{R}^N)$, as $\varepsilon \to 0$.

Now we prove that u_k is a solution of (P). Since $\hat{w}_{k,\varepsilon} \in \mathcal{M}_\varepsilon$ is a critical point for J_ε, we have

$$\int_{\mathbb{R}^N} \nabla\hat{w}_{k,\varepsilon}\nabla\phi + \frac{V(x)}{\varepsilon^2}\hat{w}_{k,\varepsilon}\phi = \lambda_{k,\varepsilon}\int_{\mathbb{R}^N} |\hat{w}_{k,\varepsilon}|^{p-1}\hat{w}_{k,\varepsilon}\phi, \qquad \forall\phi \in \mathrm{H}^1(\mathbb{R}^N), \tag{4.7}$$

where $\lambda_{k,\varepsilon} = 2c_{k,\varepsilon}$ is the Lagrange multiplier. Then, since

$$\int_{\mathbb{R}^N} \frac{V(x)}{\varepsilon^2}\hat{w}_{k,\varepsilon}\phi = 0, \qquad \text{for all } \phi \in \mathrm{C}_0^\infty(\Omega),$$

passing to the limit when $\varepsilon \to 0$, we have

$$\int_\Omega \nabla\hat{u}_k\nabla\phi = \lambda_k\int_\Omega |\hat{u}_k|^{p-1}\hat{u}_k\phi, \qquad \forall\phi \in \mathrm{C}_0^\infty(\Omega), \tag{4.8}$$

where $\lambda_k = 2c_k$. Here we have used the fact that $\hat{w}_{k,\varepsilon}$ sub-converge in $\mathrm{L}^{p+1}(\mathbb{R}^N)$ to $\hat{u}_k$, which comes from Lemma 3.1 and the compactness of the embedding $H_\varepsilon \subset \mathrm{L}^{p+1}(\mathbb{R}^N)$.

Considering (4.8) and [4, Proposition IX.18], we would be done if we proved that

$$\hat{u}_k(x) = 0, \qquad \text{a.e. } \mathbb{R}^N \setminus \Omega. \tag{4.9}$$

In fact, $\hat{u}_k|_\Omega \in \mathrm{H}_0^1(\Omega)$ would hold, and from (4.8), $J(\hat{u}_k|_\Omega) = c_k$.

Let us prove (4.9). We associate to each $\delta > 0$,

$$\varepsilon_\delta^* = \min\left\{\varepsilon_\delta, \frac{V_\delta}{(2c_k)^{1/2}}\right\}. \tag{4.10}$$

For every $(\delta,\alpha) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$ we write $S_{\delta,\alpha} = \{x \in \mathbb{R}^N \setminus \Omega^\delta : |\hat{u}_k(x)| \geq \alpha\}$. Let us assume that there exist $\delta_*, \alpha_*, \eta > 0$ such that $|S_{\delta_*,\alpha_*}| \geq \eta > 0$. Then, as $S_{\delta_*,\alpha_*} \subset S_{\delta,\alpha_*}$, for all $\delta \in (0,\delta_*)$ we have

$$|S_{\delta,\alpha_*}| \geq \eta > 0, \qquad \forall\delta \in (0,\delta_*). \tag{4.11}$$

Considering (V3), we obtain $\delta' \in (0,\delta_*)$ such that

$$V_\delta < \frac{\alpha_*^2\eta}{2}, \qquad \forall\delta \in (0,\delta'). \tag{4.12}$$

Let $\delta_0 \in (0, \delta')$ be fixed, then we have that

$$\int_{S_{\delta_0,\alpha_*}} |\hat{u}_k|^2 \geq \alpha_*^2 \eta. \tag{4.13}$$

On the other hand, for every $\sigma > 0$ there exists a $\varepsilon_\sigma \in (0, \varepsilon_\delta^*)$ such that

$$\|\hat{u}_k\|^2_{L^2(S_{\delta_0,\alpha_*})} \leq \|\hat{w}_{k,\varepsilon}\|^2_{L^2(S_{\delta_0,\alpha_*})} + \sigma, \qquad \forall \varepsilon \in (0, \varepsilon_\sigma).$$

Thus, for $\sigma = \frac{\alpha_*^2 \eta}{3}$ and $\varepsilon \in (0, \varepsilon_\sigma[$, using (4.6), (4.10) and (4.12), we get

$$\int_{S_{\delta_0,\alpha_*}} |\hat{u}_k|^2 \leq \sigma + \int_{S_{\delta_0,\alpha_*}} |w_{k,\varepsilon}|^2 \leq \frac{\alpha_*^2 \eta}{3} + \left(\frac{2c_k}{V_\delta}\right) \varepsilon^2 < \frac{\alpha_*^2 \eta}{3} + V_\delta < \frac{5}{6} \alpha_* \eta, \tag{4.14}$$

which contradicts (4.13). Hence, $|S_{\delta,\alpha}| = 0$, for all $(\alpha, \delta) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$, that is, we proved (4.9). □

Actually we have strong convergence as the following lemma asserts.

Lemma 4.2. *For every $k \in \mathbb{N}$, as $\varepsilon \to 0$, $w_{k,\varepsilon}$ sub-converge in the norm of $\mathrm{H}^1(\mathbb{R}^N)$ to u_k.*

Proof. From the compactness of the embedding $H_\varepsilon \subset \mathrm{L}^2(\mathbb{R}^N)$, it follows that $\hat{w}_{k,\varepsilon}$ sub-converge in $\mathrm{L}^2(\mathbb{R}^N)$ to $\hat{u}_k$ as $\varepsilon \to 0$; so

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |\hat{w}_{k,\varepsilon}|^2 = \int_\Omega |\hat{u}_k|^2.$$

This and (4.3) let us show that

$$\limsup_{\varepsilon \to 0} \|\hat{w}_{k,\varepsilon}\|_{H^1(\mathbb{R}^N)} \leq \|\hat{u}_k\|_{H^1(\mathbb{R}^N)},$$

concluding the proof. □

Our next goal is to obtain an exponential control of the decay of the family $\{w_{k,\varepsilon}\}$ outside Ω. For this purpose we obtain a general L^∞ estimate for solutions of an elliptic inequality, following the Moser iteration technique. We have

Proposition 4.1. *Let $D \subset \mathbb{R}^N$ be open and connected. If w is a classical solution of the elliptic inequality*

$$\begin{cases} \Delta w - f(w) \geq 0 & \text{in } D, \\ w > 0 & \text{in } D, \\ w = 0 & \text{on } \partial D, \end{cases} \tag{4.15}$$

where $N \geq 3$, $p + 1 \in (2, 2^*)$ and f satisfies

$$t f(t) \leq c t^{p+1}, \qquad \forall t \in \mathbb{R}^+, \tag{4.16}$$

for some constant $c > 0$ and if moreover $w \in \mathrm{H}_0^1(D)$ then there exists a constant $C = C(c, p, N) > 0$ such that

$$\|w\|_{L^\infty(D)} \leq C \|w\|^{4/[(N+2)-p(N-2)]}_{L^{2^*}(D)}. \tag{4.17}$$

This result was proved in [5] assuming that $D \subset \mathbb{R}^N$ is smooth and bounded. It can be extended to a not necessarily bounded D or regular ∂D. We can follow the step in [5], by choosing a slightly modified test function depending on a parameter, in order to avoid the possible non-regularity of the boundary. We omit the details.

Lemma 4.3. *For every $k \in \mathbb{N}$ there exists a $K_2 > 0$ such that*

$$\|w_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} < K_2, \qquad \forall \varepsilon \in (0, \varepsilon_\delta). \tag{4.18}$$

Proof. Given $\varepsilon \in (0, \varepsilon_\delta)$, we consider D_ε^+ a connected component of the set $\{x \in \mathbb{R}^N : w_{k,\varepsilon} > 0\}$. So, we have

$$\begin{cases} \Delta w_{k,\varepsilon} + w_{k,\varepsilon}^p \geq 0 & \text{in } D_\varepsilon^+, \\ w_{k,\varepsilon} > 0 & \text{in } D_\varepsilon^+, \\ w_{k,\varepsilon} = 0 & \text{on } \partial D_\varepsilon^+, \end{cases} \tag{4.19}$$

hence, from (4.4) and Proposition 4.1,

$$\|w_{k,\varepsilon}\|_{L^\infty(D_\varepsilon^+)} \leq K_2, \qquad \forall \varepsilon \in (0, \varepsilon_\delta), \tag{4.20}$$

where the constant K_2 depends on N, k and p. Since D_ε^+ is arbitrary, the inequality holds in $\{x \in \mathbb{R}^N : w_{k,\varepsilon} > 0\}$. By a similar argument we also show that the inequality holds in $\{x \in \mathbb{R}^N : w_{k,\varepsilon} < 0\}$. □

Remark 4.1. Since $v_{k,\varepsilon} = \varepsilon^{2/(p-1)} w_{k,\varepsilon}$, it follows from Lemma 4.3 that

$$\lim_{\varepsilon \to 0} \|v_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} = 0, \qquad \forall k \in \mathbb{N}. \tag{4.21}$$

Moreover, since $\|u_k\|_{L^{p+1}(\mathbb{R}^N)} \neq 0$ for all $k \in \mathbb{N}$, it is clear that there exists a constant $\eta_k > 0$ such that

$$\liminf_{\varepsilon \to 0} \frac{\|v_{k,\varepsilon}\|_{L^\infty(\mathbb{R}^N)}}{\varepsilon^{2/(p-1)}} \geq \eta_k > 0. \tag{4.22}$$

In order to obtain the exponential decay of $w_{k,\varepsilon}$, we shall give a comparison argument as in [6]. We consider a positive solution for the problem

$$\begin{cases} \Delta U - 2bU = 0 & \text{in } \mathbb{R}^N \setminus \Omega^\delta, \\ U = a & \text{on } \partial \Omega^\delta, \\ \lim_{|x| \to \infty} U(x) = 0, & \end{cases} \tag{4.23}$$

where $a, b > 0$. Such a solution satisfies

$$U(x) \leq C \exp\{-b \cdot \operatorname{dist}(x, \Omega^\delta)\}, \qquad \forall x \in \mathbb{R}^N \setminus \Omega^\delta \tag{4.24}$$

for some constant C depending on a and Ω^δ, see [6, Lemma 2.7].

Lemma 4.4. *For every $k \in \mathbb{N}$, $\delta, c > 0$, there exists $\varepsilon_{**} \in (0, \varepsilon_\delta)$ such that*

$$|w_{k,\varepsilon}(x)| < C \cdot \exp\left\{-\frac{c}{\varepsilon} \cdot \operatorname{dist}(x, \Omega^\delta)\right\}, \qquad \forall x \in \mathbb{R}^N, \forall \varepsilon \in (0, \varepsilon_{**}), \tag{4.25}$$

where C depends on K_2 and Ω^δ.

Proof. Let $\varepsilon_* \in (0, \varepsilon_\delta)$ such that $V_\delta > (K_2 + 2c/\varepsilon_*)\varepsilon_*^2$. Then, from Lemma 4.3 and for all $\varepsilon \in (0, \varepsilon_*)$ and $x \in \mathbb{R}^N \setminus \overline{\Omega^\delta}$, we have that

$$F_{k,\varepsilon}(x) \equiv \frac{V(x)}{\varepsilon^2} - |w_{k,\varepsilon}|^{p-1} \geq \frac{V_\delta}{\varepsilon^2} - K_2 > 2\frac{c}{\varepsilon}.$$

Now we consider U a solution to problem (4.23) with $a = K_2$ and $b = c/\varepsilon$. Then,

$$\begin{cases} \Delta U - F_{k,\varepsilon}(x)U \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega^\delta, \\ U = K_2 & \text{on } \partial\Omega^\delta, \\ \lim_{|x|\to\infty} U(x) = 0, \end{cases} \tag{4.26}$$

from where it follows that

$$\begin{cases} \Delta(U - w_{k,\varepsilon}) - F_{k,\varepsilon}(x)(U - w_{k,\varepsilon}) \leq 0 & \text{in } \mathbb{R}^N \setminus \Omega^\delta, \\ U - w_{k,\varepsilon} > 0 & \text{on } \partial\Omega^\delta, \\ \lim_{|x|\to\infty} (U(x) - w_{k,\varepsilon}(x)) = 0. \end{cases} \tag{4.27}$$

Now it is clear that

$$w_{k,\varepsilon}(x) \leq U(x), \qquad \forall x \in \mathbb{R}^N \setminus \Omega^\delta.$$

Analogously we can prove that

$$-U(x) \leq w_{k,\varepsilon}(x), \qquad \forall x \in \mathbb{R}^N \setminus \Omega^\delta.$$

Then, using (4.24) we obtain

$$|w_{k,\varepsilon}(x)| \leq C \exp\left\{-\frac{c}{\varepsilon}\operatorname{dist}(x, \Omega^\delta)\right\}, \qquad \forall x \in \mathbb{R}^N \setminus \Omega^\delta,$$

and enlarging C is necessary, we finally get the inequality in all $\mathbb{R}^N$. □

5. Asymptotic behavior at the boundary

We already know that the sequence $w_{k,\varepsilon}$ converges in $\mathrm{H}^1(\mathbb{R})$ to a function u which is a solution of (P) in Ω. By elliptic regularity it is not hard to prove that on each compact set $D \subset \Omega$, the convergence of $w_{k,\varepsilon}$ to u is uniform on D. On the other hand, outside Ω, namely in Ω^δ, we have exponential decay according to Lemma 4.4. The uniform behavior of $w_{k,\varepsilon}$ on the boundary of Ω is not covered by these two arguments. In this section we prove

Proposition 5.1. *The family of solutions $w_{k,\varepsilon}$ verifies*

$$\lim_{\varepsilon\to 0} \max_{x\in\partial\Omega} |w_{k,\varepsilon}(x)| = 0, \qquad \forall k \in \mathbb{N}. \tag{5.1}$$

For proving this proposition we see two preliminary lemmas. Let $\delta > 0$ be small enough that the set $\Omega_\delta \equiv \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$ is not empty. We define the ring around $\partial\Omega$ as $R(\delta) = \Omega^\delta \setminus \overline{\Omega_\delta}$ and we consider $M_\varepsilon(\delta) = \max_{x\in\partial R(\delta)} |w_{k,\varepsilon}(x)|$ for $k \in \mathbb{N}$ fixed. First we show that

Lemma 5.1. *Given $\sigma > 0$, there exists $\delta_\sigma > 0$ such that*

$$\max_{x\in R(\delta)} |w_{k,\varepsilon}(x)| \leq M_\varepsilon(\delta) + \sigma, \qquad \forall \delta \in (0, \delta_\sigma). \tag{5.2}$$

Proof. For notational convenience we write $w = w_{k,\varepsilon}$ and $R^{\pm}(\delta) = \{x \in R(\delta) : \pm w > 0\}$. Then we have

$$\begin{cases} \pm\Delta w \pm |w|^{p-1}w \geq 0, & \text{in } R^{\pm}(\delta), \\ \pm w \geq 0, & \text{on } \partial R^{\pm}(\delta). \end{cases} \tag{$D^{\pm}$}$$

We consider only (D^+) since the other case is analogous. We put $v = w - M_\varepsilon(\delta)$ to get

$$\begin{cases} \Delta v \geq f, & \text{in } R^+(\delta), \\ v \leq 0, & \text{on } \partial R^+(\delta) \end{cases} \tag{5.3}$$

where $f \equiv -|w|^{p-1}(M_\varepsilon(\delta) + v)$. Then, using the Alexandroff Maximum Principle ([18, Theorem 2.21]), we obtain

$$\sup_{R^+(\delta)} v \leq C \cdot \| f^- \|_{L^N(R^+(\delta))} \leq C|R^+(\delta)|^{1/N} K_2^{p-1}(M_\varepsilon(\delta) + \sup_{R^+(\delta)} v),$$

where $C = C(N, \text{diam}(\Omega)) > 0$. Now choosing $\delta_\sigma > 0$ small enough, we get

$$\sup_{R^+(\delta)} w \leq M_\varepsilon(\delta) + \sigma, \qquad \forall \delta \in (0, \delta_\sigma). \tag{5.4}$$

In a similar way, decreasing δ_σ if necessary, we find also

$$\inf_{R^-(\delta)} w \geq -M_\varepsilon(\delta) - \sigma, \qquad \forall \delta \in (0, \delta_\sigma), \tag{5.5}$$

completing the proof of the lemma. □

Next we control the values of w on $\partial R(\delta)$, that is

Lemma 5.2. *Given $\sigma > 0$, there exist $\delta', \varepsilon' > 0$ such that*

$$M_\varepsilon(\delta') < \sigma, \qquad \forall \varepsilon \in (0, \varepsilon'). \tag{5.6}$$

We observe that with this lemma and Lemma 5.1 we can complete the proof of Proposition 5.1. In fact $\partial\Omega \subset R(\delta)$ and so $\max_{x\in\partial\Omega} |w_{k,\varepsilon}(x)| \leq \max_{x\in R(\delta)} |w_{k,\varepsilon}(x)|$.

Proof of Lemma 5.2. Denoting

$$m_\delta(\varepsilon) = \max_{x\in\partial\Omega_\delta} |w_{k,\varepsilon}(x)| \qquad \text{and} \qquad m^\delta(\varepsilon) = \max_{x\in\partial\Omega^\delta} |w_{k,\varepsilon}(x)|,$$

we see that we need to show that $m_\delta(\varepsilon)$ and $m^\delta(\varepsilon)$ are controlled by σ. First, we see that

$$\lim_{\varepsilon\longrightarrow 0} m^\delta(\varepsilon) = 0.$$

In fact, outside $\Omega^{\delta/2}$, $w_{k,\varepsilon}$ decay exponentially, as proved in Lemma 4.4; then $w_{k,\varepsilon} \to 0$ uniformly in $\partial\Omega^\delta$. Second, we study $m_\delta(\varepsilon)$. We denote by K_{c_k} the set of critical points of the functional J corresponding to the critical value c_k. According to Lemmas 4.1 and 4.2, there exists $u \in K_{c_k}$ and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $w_{k,\varepsilon_n} \equiv w_n \to u$ in $\mathrm{H}^1(\mathbb{R}^N)$ and pointwise. We choose $\eta > 0$ such that $R_\delta(\eta) = \Omega_{\delta-\eta} \setminus \overline{\Omega_{\delta+\eta}}$ verifies $R_\delta(\eta) \cap \partial\Omega = \emptyset$. From elliptic estimates, we see that for each compact set $D \subset \Omega$, the convergence of w_n to u is uniform in D. Then, in particular, given $\sigma > 0$, there exists an $n^* = n^*(\sigma, w) \in \mathbb{N}$ such that

$$\max_{x\in\overline{R_\delta(\eta)}} |w_n(x) - u(x)| < \frac{\sigma}{2}, \qquad \forall n > n^*. \tag{5.7}$$

On other hand, since $u|_{\mathbb{R}^N\setminus\overline{\Omega}} = 0$ and u is a solution of (P), there exists a $\delta' = \delta'(\sigma, w) > 0$ such that

$$\max_{x\in\overline{R_{\delta'}(\eta)}} |u(x)| < \frac{\sigma}{2}. \tag{5.8}$$

Then, from (5.7) and (5.8), we get

$$m_{\delta'}(\varepsilon_n) < \sigma, \qquad \forall n > n^*. \tag{5.9}$$

We see that the values n^* and δ' may depend on u. However, one can argue using the compactness of the set K_{c_k}, that they can be chosen so they actually depend only on k, but not on the particular $u \in K_{c_k}$. □

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