PDEs in moving time dependent domains *

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Dedicated to Professor M.G. Velarde
in occasion of his 70th birthdate.

Abstract

In this work we study partial differential equations defined in a domain that moves in time according to the flow of a given ordinary differential equation, starting out of a given initial domain. We first derive a formulation for a particular case of partial differential equations known as balance equations. For this kind of equations we find the equivalent partial differential equations in the initial domain and later we study some particular cases with and without diffusion. We also analyze general second order differential equations, not necessarily of balance type. The equations without diffusion are solved using the characteristics method. We also prove that the diffusion equations, endowed with Dirichlet boundary conditions and initial data, are well posed in the moving domain. For this we show that the principal part of the equivalent equation in the initial domain is uniformly elliptic. We then prove a version of the weak maximum principle for an equation in a moving domain. Finally we perform suitable energy estimates in the moving domain and give sufficient conditions for the solution to converge to zero as time goes to infinity

1 Introduction

In a standard setting for many partial differential equations of mathematical physics, one usually assumes that the physical process being described occurs in a fixed domain of the physical space. This includes many equations describing the motion of fluids for example,

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despite the fact that particle fluids and hence fluid subdomains actually move with time. Of course there are some other problems,such as free boundary problems, in which the physical domain of the PDE changes with time. In all these problems the motion of particles or subdomains occurs according to an unknown velocity field with is actually one of the main unknowns of the problem.

In this paper we assume some intermediary situation in which each point of a given initial domain $\Omega_0 \subset \mathbb{R}^n$, moves in time according to some prescribed autonomous vector field. Hence at later times the domain Ω_0 evolves into a diffeomorphic domain $\Omega(t)$ (which is not excluded to coincide with Ω_0 itself!). In particular, topological properties of the domain are preserved along time. However the geometrical evolution of the domain can be very complex; for example one can consider the evolution of the open set Ω_0 in $\mathbb{R}^3$ with the vector field of the Lorenz equations in a chaotic regime.

Our goal is the to describe some sensible class of PDEs to be consider in such a family of moving domains. We choose then to describe balance equations in moving domains, which result from conservation principles and which have natural applications to conservation of mass, momentum, energy etc. For such equations one must then give some suitable definition of solution.

After giving a convenient meaning of solution for both balance and general parabolic equations, we prove that such equations can be solved using available results.

Then we investigate, on some particular, although significative examples of equations in moving domains, basic tools in the analysis of parabolic equations such as the (weak) maximum principle and energy estimates. In particular we obtain sufficient conditions on the equations and on the moving domains, that guarantee that the solutions converge to zero as time goes to infinity.

2 Moving domains

We assume that each point x of an original given domain (smooth open set) $\Omega_0 \subset \mathbb{R}^n$, starting at time $t = 0$ moves following a curve $t \longmapsto Y(t;x)$, in $\mathbb{R}^n$. Moreover we assume this curve is a solution of the autonomous system of ODEs

$$\begin{cases} \dot{Y}(t;x) = \vec{V}(Y(t;x)) \\ Y(0;x) = x \end{cases} \tag{2.1}$$

for some given smooth vector velocity field $\vec{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Even more and for simplicity we assume that all solutions of (2.1) are defined for all $t \in \mathbb{R}$.

Hence, for $t \in \mathbb{R}$, we have a deformation map

$$\phi(t) : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \phi(t)z = Y(t;z)$$

which is a diffeormorphism that satisfies the group properties $\phi(0) = I$, $\phi(t+s) = \phi(t) \circ \phi(s)$ for all $t, s \in \mathbb{R}$. In particular $\phi(-t)$ is the inverse of $\phi(t)$.

Therefore, the original domain Ω_0 is deformed into the domains

$$\Omega(t) = \phi(t)\Omega_0 \qquad t \in \mathbb{R}$$

and the boundaries satisfy $\partial\Omega(t) = \phi(t)\ \partial\Omega_0$. Also, any smooth subdomain W_0 of Ω_0 is also deformed into

$$W(t) = \phi(t) W_0, \quad t \in \mathbb{R}$$

and its boundary is given by $\partial W(t) = \phi(t)\partial W_0$.

The next results gives geometrical information about the deformations above.

Lemma 2.1 *With the above notations, for $x_0 \in \partial\Omega_0$ then $\phi(t)x_0 \in \Omega(t)$ and*

$$D\phi(t)(x_0)$$

is an isomorphism in $\mathbb{R}^n$ that transforms the tangent plane in $x_0 \in \partial\Omega_0$, that we denote $T_{x_0}\partial\Omega_0$, into the tangent plane to $\partial\Omega(t)$ at $\phi(t)x_0$, $T_{\phi(t)x_0}\partial\Omega(t)$.

Proof. Just note that if $z(s)$ is a curve in $\partial\Omega_0$ with $z(0) = x_0$, then $z'(0) = v_0$ is a tangent vector at x_0 (and conversely). Hence, $w(s) = \phi(t)(z(s))$ is a curve in $\partial\Omega(t)$, with $w(0) = y_0$ and

$$w'(0) = D\phi(t)(x_0)v_0$$

is a tangent vector at $\partial\Omega(t)$. ■

We also recall the following

Definition 2.2 *A matrix $\eta(t)$ is a fundamental matrix of the linear system*

$$X'(t) = A(t)X(t) \tag{2.2}$$

iff each column of $\eta(t)$ is a solution of (2.2) and $\eta(t)$ is nonsingular.

Observe that in particular, $\eta'(t) = A(t)\eta(t)$. Then we have

Lemma 2.3 *If $\eta(t)$ is a fundamental matrix of (2.2), then*

$$\gamma(t) = \left(\eta^{-1}(t)\right)^* = \left(\eta^*(t)\right)^{-1}$$

is a fundamental matrix of the adjoint system

$$Y'(t) = -A^*(t)Y(t)$$

*where * denotes the adjoint matrix.*

Proof. Differentiate in

$$\eta^{-1}(t) \circ \eta(t) = I$$

and use (2.2). ■

The following result is obtained from classical results in ODEs, see [4].

Proposition 2.4
i) For $x \in \mathbb{R}^n$, $D\phi(t)x$ is a fundamental matrix of

$$\dot{Z}(t) = A(t)Z(t)$$

and $D\phi(0) = I$, where $A(t) = D\vec{V}(\phi(t)x)$.
ii) Denote

$$|K(x,t)| = det(D\phi(t)x), \quad x \in \mathbb{R}^n!`$$

*then we have the **Abel–Liouville–Jacobi** formula*

$$\frac{\partial}{\partial t}|K(x,t)| = tr\big(D\vec{V}\big)(\phi(t)x)\,|K(x,t)| = div\big(\vec{V}\big)(\phi(t)x)\,|K(x,t)|$$

hence

$$|K(x,t)| = e^{\int_0^t div\vec{V}(\phi(s)x)\,ds}.$$

In particular, for $t \in [-T,T]$ there exist $C_1(T), C_2(T)$ such that

$$0 < C_1(T) \le |K(x,t)| \le C_2(T) \quad \forall x \in \Omega_0 \ \ \forall t \in [0,T]. \tag{2.3}$$

Remark 2.5 *Observe that if $W_0 \subset \Omega_0$ and $W(t) = \phi(t)W_0$ then the measure of $W(t)$ satisfies*

$$|W(t)| = \int_{W(t)} 1\,dy = \int_{W_0} |K(x,t)|\,dx = \int_{W_0} e^{\int_0^t div\vec{V}(\phi(s)(x))\,ds}\,dx.$$

In particular, if $div(\vec{V}) = 0$ then the measure is preserved, that is,

$$|W(t)| = |W_0| \qquad \forall\ W_0 \subset \Omega_0 \quad \forall t \in \mathbb{R}.$$

Also, if $div(\vec{V}) \le -d_0 < 0$ at every point, then

$$|W(t)| \le |W_0| \quad e^{-d_0 t}$$

and we say the flow of (2.1) is contractive.

Finally if $div(\vec{V}) \ge d_0 > 0$ at every point, then

$$|W(t)| \ge |W_0| \quad e^{d_0 t}$$

and we say the flow if expansive.

For example for a linear flow, that is, $\vec{V}(x) = Mx$ for a given matrix M, we have

$$div(\vec{V}) = tr(M) = \sum_{i=1}^{n} \mu_i = d_0$$

is the trace of M, that is the sum of all eigenvalues of M.

Then we have the following result that complements Lemma 2.1.

Corollary 2.6 *Assume $x_0 \in \partial\Omega_0$ and consider $y_0 = \phi(t)x_0 \in \partial\Omega(t)$. Then if $\vec{n}(x_0)$ is an unitary outward normal vector to Ω_0 at x_0 then*

$$N(y_0) = ((D\phi(t)x_0)^*)^{-1}\vec{n}(x_0)$$

is an outward vector at y_0. That is, $((D\phi(t))^*x_0)^{-1}$ is a linear isomorphism in $\mathbb{R}^n$ that transforms the normal space at $x_0 \in \partial\Omega_0$, which we denote, N_{x_0}, into the normal space to $\Omega(t)$ at $y_0 \in \partial\Omega(t)$, which we denote N_{y_0}.

Proof. From Lemma 2.1 a normal vector at $y_0 = \phi(t)(x_0) \in \partial\Omega(t)$, $\vec{n}$, must satisfty

$$< \vec{n}, D\phi(t)x_0\vec{\tau} >= 0 \quad \forall \vec{\tau} \in T_{x_0}\partial\Omega_0$$

which reads

$$< (D\phi(t)x_0)^*\vec{n}, \vec{\tau} >= 0 \quad \forall \vec{\tau} \in T_{x_0}\partial\Omega_0.$$

Hence we can take $\vec{n}$ such that $((D\phi(t)x_0)^*)\vec{n} = \vec{n}(x_0)$ which gives the result. ■

3 Balance equations

The following notations will be used throughout the paper.

Definition 3.1 *If for some $T > 0$, f is defined in*

$$f : \cup_{t\in(-T,T)}\Omega(t) \times \{t\} \longrightarrow \mathbb{R}, \quad (y,t) \longmapsto f(y,t)$$

then we define $\overline{f}$ in Ω_0 as

$$\overline{f} : \Omega_0 \times (-T,T) \longrightarrow \mathbb{R}, \quad \overline{f}(x,t) = f(\phi(t)x,t)$$

Consider $W(t) = \phi(t)W_0 \subset \Omega(t)$, a sufficiently smooth region with boundary $\partial W(t)$. Then the time variation of the amount of T in $W(t)$ is given by

$$\frac{d}{dt}\int_{W(t)} T(y,t)\, dy$$

which is computed below. Note that this is the classical Reynolds Transport theorem, [6, 3, 5].

Proposition 3.2 *With the notations above, we have that*

$$\frac{d}{dt}\int_{W(t)} T(y,t)\, dy$$

can be written by either one of the following equivalent expressions

$$\int_{W_0} \frac{\partial\, \overline{T}}{\partial t}(x,t)\, |K(x,t)|\, dx + \int_{W_0} \overline{T}(x,t)\, \overline{div\vec{V}}(x,t)\, |K(x,t)|\, dx \tag{3.1}$$

or

$$\int_{W(t)} \frac{\partial T}{\partial t}(y,t)\,dy + \int_{W(t)} div_y(T(y,t)\ .\ \vec{V}(y))\,dy \tag{3.2}$$

or

$$\int_{W(t)} \frac{\partial T}{\partial t}(y,t)\,dy + \int_{\partial W(t)} T(y,t)\vec{V}(y)\,d\vec{s}. \tag{3.3}$$

Now we will derive the Balance Equations for the quantity $T(y,t)$. In fact we have

$$\frac{d}{dt}\int_{W(t)} T(y,t)\,dy = \int_{W(t)} f(y,t)\,dy - \int_{\partial W(t)} \vec{J}\,d\vec{s}$$

where $f(y,t)$ represents the rate of production/consumption of T per unit volume in $W(t)$ and $\vec{J}$ is the vector field of the flow of T across the boundary of $W(t)$. Hence the divercence theorem leads to

$$\frac{d}{dt}\int_{W(t)} T(y,t)\,dy = \int_{W(t)} f(y,t)\,dy - \int_{W(t)} div_y\vec{J}\,dy \tag{3.4}$$

Hence, (3.4) and the Proposition above leads to

Proposition 3.3 *Under the assumptions and notations above, the magnitud T satisfies the balance equations in the moving domains, if and only if the following equivalent conditions are satisfied:*

$$\frac{\partial T}{\partial t}(y,t) + div_y(T(y,t)\ .\ \vec{V}(y)) = f(y,t) - div_y(\vec{J}), \quad y \in \Omega(t),\ t > 0 \tag{3.5}$$

or

$$\frac{\partial}{\partial t}\overline{T}(x,t) + \overline{T}(x,t)\ \ \overline{div(\vec{V})}(x,t) = \overline{f}(x,t) - \overline{div_y(\vec{J})}(x,t) \quad x \in \Omega_0,\ t > 0. \tag{3.6}$$

Proof. First, equating (3.2) and (3.4) we get

$$\int_{W(t)} \left(\frac{\partial T}{\partial t}(y,t) + div_y(T(y,t)\vec{V}(y))\right)\,dy = \int_{W(t)} (f(y,t) - div_y\vec{J})\,dy.$$

Since $W(t) = \phi(t)(W_0)$, $\phi(t)$ is a diffeormorphism and W_0 is arbitrary, we get (3.5).

Now, using $y = \phi(t)x$ we get in the right hand side of (3.4)

$$\int_{W_0} \overline{f}(x,t)\,|K(x,t)|\ dx - \int_{W_0} \overline{div_y(\vec{J})}(x,t)\,|K(x,t)|\ dx,$$

equating to (3.1) and using that W_0 is arbitrary, we get (3.6). ■

4 Boundary and initial conditions

As we consider Dirichlet boundary conditions and using

$$y = \phi(t)x, \qquad \Omega(t) = \phi(t)\Omega_0, \qquad \partial\Omega(t) = \phi(t)(\partial\Omega_0)$$

then

$$T(y,t) = 0 \quad \forall\, y \in \partial\Omega(t) \quad \Leftrightarrow \quad \overline{T}(x,t) = 0 \quad \forall\, x \in \partial\Omega_0$$

As for the initial condition we have, since $\phi(0) = I$,

$$T(y,0) = T_0(y) \;\; \forall\, y \in \Omega_0 \quad \Leftrightarrow \quad \overline{T}(x,0) = T_0(x) \;\; \forall\, x \in \Omega_0.$$

Thus, (3.5) and (3.6), with boundary and initial conditions read, respectively,

$$\begin{cases} \frac{\partial T}{\partial t}(y,t) + div_y(T(y,t)\ .\ \vec{V}(y)) = f(y,t) - div_y(\vec{J}) & y \in \Omega(t) \\ T(y,t) = 0 \;\; y \in \partial\Omega(t) \quad \forall t \qquad T(y,0) = T_0(y) & y \in \Omega_0 \end{cases} \tag{4.1}$$

$$\begin{cases} \frac{\partial}{\partial t}\overline{T}(x,t) + \overline{T}(x,t)\ \ \overline{div(\vec{V})}(x,t) = \overline{f}(x,t) - \overline{div_y(\vec{J})}(x,t) & x \in \Omega_0 \\ \overline{T}(x,t) = 0 \;\; x \in \partial\Omega_0 \quad \forall t \qquad \overline{T}(x,0) = T_0(x) & x \in \Omega_0. \end{cases} \tag{4.2}$$

In fact we use (4.2) to define a solution of (4.1), i.e.

$$T(y,t) \;\; \text{satisfies} \;\; (4.1) \quad \Leftrightarrow \overline{T}(x,t) \;\; \text{satisfies} \;\; (4.2).$$

5 Balance equations without diffusion

5.1 No flux and no diffusion: pure inertia

With the previous notations, assume $div_y(\vec{J}) = 0$ then the following problems are equivalent

$$\begin{cases} \frac{\partial}{\partial t}T(y,t) + div_y(T(y,t)\ .\ \vec{V}(y)) = f(y,t) & y \in \Omega(t) \\ T(y,t) = 0 \; y \in \partial\Omega(t) \;\; \forall t \qquad T(y,0) = T_0(y) & y \in \Omega_0 \end{cases} \tag{5.1}$$

and

$$\begin{cases} \frac{\partial}{\partial t}\overline{T}(x,t) + \overline{T}(x,t)\ \ \overline{div(\vec{V})}(x,t) = \overline{f}(x,t) & x \in \Omega_0 \\ \overline{T}(x,t) = 0 \quad x \in \partial\Omega_0 \quad \forall t \qquad \overline{T}(x,0) = T_0(x) & x \in \Omega_0. \end{cases} \tag{5.2}$$

Then we have

Proposition 5.1 *With the notations above, (5.1) and (5.2) have a unique explicit solution given by*

$$T(y,t) = T_0(x)\ e^{-\int_0^t div_y \vec{V}(\phi(r)x)\,dr}\ + \int_0^t\ e^{-\int_s^t div_y \vec{V}(\phi(r)x)\,dr}\ f(y,s)\,ds, \quad y = \phi(t)x \in \Omega(t)$$

and

$$\overline{T}(x,t) = T_0(x)\ e^{-\int_0^t div \vec{V}(\phi(r)x)\,dr}\ + \int_0^t\ e^{-\int_s^t div \vec{V}(\phi(r)x)\,dr}\ \overline{f}(x,s)\,ds, \quad x \in \Omega_0,$$

respectively.

Proof. The solution of (5.2) is obtained by solving a linear nonhomogeneous ODE

$$Z'(t) + P(t)Z(t) = h(t), \quad Z(0) = Z_0$$

for each $x \in \Omega_0$. From this the solution of (5.1) is immediate. ■

Remark 5.2 *Assume in particular that there are no source terms, that is, $f = 0$. Hence in (5.1) we have*

$$T(y,t) = T_0(x)\ e^{-\int_0^t div_y \vec{V}(\phi(r)x)\,dr}, \quad y = \phi(t)x$$

Thus, if moreover $div(\vec{V}) = 0$ then

$$T(y,t) = T_0(x) \quad y = \phi(t)x,$$

and T remains constant along the paths of the flow.

On the other hand if the flow is expansive then $T(y,t)$ decreases along the paths of the flow, while it increases if the flow is contractive.

5.2 Flux and no diffusion: transport equations

Below we use $\psi(t) = \phi^{-1}(t) = \phi(-t)$.

Proposition 5.3 *If we assume*

$$\vec{J}(y,t) = \vec{a}(y,t)T(y,t) \qquad y \in \ \Omega(t)$$

with a C^1 scalar field

$$\vec{a} : \mathbb{R}^n \times \mathbb{R} \ \ \rightarrow \ \ \mathbb{R}$$

then the balance equations (4.1) and (4.2) read

$$\begin{cases} \frac{\partial}{\partial t}T(y,t) + div_y(T(y,t)\ .\ \vec{V}(y)) + \nabla_y T(y,t).\vec{a}(y,t) + div_y(\vec{a}(y,t))T(y,t) = f(y,t) \quad y\ \in \Omega(t) \\ T(y,t) = 0 \quad y \in \partial\Omega(t)\ \ \forall t \qquad\qquad T(y,0) = T_0(y) \qquad y \in \Omega_0 \end{cases} \tag{5.3}$$

and

$$\begin{cases} \frac{\partial \overline{T}}{\partial t}(x,t) + \overline{T}(x,t)C(x,t) + \nabla_x \overline{T}(x,t)\vec{b}(x,t) = \overline{f}(x,t) \quad x \in \Omega_0 \\ \\ \overline{T}(x,t) = 0 \ x \in \partial\Omega_0 \ \ \forall t \qquad \overline{T}(x,0) = T_0(x) \quad x \in \Omega_0 \end{cases} \tag{5.4}$$

which are equivalent, where

$$C(x,t) = \overline{div_y(\vec{V})}(x,t) + \overline{div_y(\vec{a})}(x,t), \quad \vec{b}(x,t) = \overline{D\psi(t)y \cdot \vec{a}(y,t)}.$$

Proof. Note that (5.3) follows by direct computation from (4.1) using

$$div_y(\vec{a}(y,t)\ T(y,t)) = \nabla_y T(y,t).\vec{a}(y) + T(y,t) div_y(\vec{a}(y,t)).$$

On the other hand, for (5.4) we have to write $div_y(a(y,t)T(y,t))$ in terms of x. For this we observe that since $x = \psi(t)y$ we have $T(y,t) = \overline{T}(\psi(t)y,t)$ and then

$$\frac{\partial T}{\partial y_i}(y,t) = \sum_{j=1}^{n} \frac{\partial \overline{T}}{\partial x_j}(x,t) \frac{\partial \psi_j(t)y}{\partial y_i} \tag{5.5}$$

and $\nabla_y T(y,t) = \nabla_x \overline{T}(x,t) D\psi(t)y$.

Thus, $\nabla_y T(y,t).\vec{a}(y,t) = \nabla_x \overline{T}(x,t)\,(D\psi(t)y\ .\ \vec{a}(y,t))$ and hence

$$\overline{\nabla_y T(y,t).\vec{a}(y,t)} = \nabla_x \overline{T}(x,t) \overline{(D\psi(t)y \cdot \vec{a}(y,t))} = \nabla_x \overline{T}(x,t)\vec{b}(x,t).$$

■

Now we show that under some natural geometrical conditions (5.4) (and hence (5.3)) can be solved by using characteristics. Note that we now disregard boundary conditions.

Proposition 5.4 *Assume that for all time and $y \in \partial\Omega(t)$, we have*

$$< \vec{a}(y,t), \vec{n_0}(y) > \leq 0$$

where $< \cdot, \cdot >$ is the scalar product and and $\vec{n_0}(y)$ is the unit outward normal vector at y.

Then (5.3) and (5.4) have a unique solution.

Proof. For (5.4) we use the method of characteristics. Hence, for $x_0 \in \Omega_0$ we define curves defined on some interval I containing 0

$$s \longmapsto X(s)\ \in \Omega_0, \quad X(0) = x_0, \qquad s \longmapsto t(s)\ \in \mathbb{R}^+, \quad t(0) = 0,$$

and $s \longmapsto Z(s) = \overline{T}(X(s), t(s))$. Then

$$\frac{d}{ds} Z(s) = \nabla_x \overline{T}(X(s), t(s)) X'(s) + \frac{\partial}{\partial t}\overline{T}(X(s), t(s)) t'(s).$$

So from (5.4) we choose

$$t'(s) = 1, \quad t(0) = 0,$$

$$X'(s) = \vec{b}(X(s), t(s)), \quad X(0) = x_0$$

which gives $t(s) = s$ and

$$X'(t) = \vec{b}(X(t), t), \quad X(0) = x_0 \in \Omega_0, \tag{5.6}$$

which has a solution because $\vec{b} \in C^1(\mathbb{R}^n)$.

Hence, from (5.6) and (5.4)

$$\begin{cases} \frac{d}{dt} Z(t) + C(X(t), t) Z(t) = \overline{f}(X(t), t) \\ Z(0) = T_0(x_0) \end{cases}$$

whose solution is given by

$$Z(t) = T_0(x_0) e^{-\int_0^t C(X(r), r)\, dr} + \int_0^t e^{-\int_s^t C(X(r), r)\, dr} \overline{f}(X(s), s)\, ds. \tag{5.7}$$

In the computation above we need the solution of (5.6) not to leave Ω_0. Thus, if $X(t)$ reaches the boundary of Ω_0 at time t_0 at the point $y_0 = x(t_0) \in \partial\Omega_0$, the tangent vector to the characteristic curve at this point is $X'(t_0) = \vec{b}(x_0, t_0)$, and therefore if it points inward, that is, if

$$< \vec{b}(x_0, t_0), \vec{n}(x_0) > \leq 0 \tag{5.8}$$

then it will remain in Ω. Note now that from (5.8)

$$< \vec{b}(x_0, t_0), \vec{n}(x_0) > = < D\psi(t_0) y_0 . \vec{a}(y_0, t_0), \vec{n}(x_0) > = < \vec{a}(y_0, t_0), (D\psi(t_0) y_0)^* \vec{n}(x_0) > =$$

$$= < \vec{a}(y_0, t_0), ((D\phi(t_0) x_0)^*)^{-1} \vec{n}(x_0) > = < \vec{a}(y_0, t_0), N(y_0) > \leq 0$$

where we have used Corollary 2.6 and the assumption of this Proposition.

With this (5.7) gives the values of the solution in the moving domain. ■

6 Balance equations with diffusion

Recalling the equivalent equations (4.1) and (4.2) we have

Proposition 6.1 *Assume the flux vector field is given by*

$$\vec{J}(y, t) = -k \nabla_y T(y, t) \qquad y \in \Omega(t)$$

for some $k > 0$, then (4.1) and (4.2) read

$$\begin{cases} \frac{\partial}{\partial t} T(y, t) + \nabla_y T(y, t) . \vec{V}(y) + T(y, t) div(\vec{V})(y) - k \Delta T(y, t) = f(y, t) \quad t \in \Omega(t) \\ T(y, t) = 0 \quad y \in \partial\Omega(t) \ \ \forall t \qquad T(y, 0) = T_0(y) \qquad y \in \Omega_0 \end{cases} \tag{6.1}$$

and

$$\begin{cases} \frac{\partial \overline{T}(x,t)}{\partial t} + \overline{T}(x,t)\ \overline{div(\vec{V})}(x,t) - k\left(\sum_{k,i=1}^{n} a_{k,i}(x,t)\frac{\partial^2\ \overline{T}(x,t)}{\partial x_k \partial x_i}\ +\ \sum_{i=1}^{n}\frac{\partial \overline{T}(x,t)}{\partial x_i}.s_i(x,t)\right) = \overline{f}(x,t) \\ \\ \overline{T}(x,t) = 0\ x \in \partial\Omega_0\ \ \forall t \qquad \overline{T(x,0)} = T_0(x) \quad x \in \Omega_0 \end{cases} \tag{6.2}$$

where

$$a_{k,i}(x,t) = \sum_{j=1}^{n}\frac{\partial \psi_k(t)y}{\partial y_j}\ .\ \frac{\partial \psi_i(t)y}{\partial y_j} = \nabla_y\psi_k.\nabla_y\psi_i\ \ ,\ y = \phi(t)x$$

and

$$s_i(x,t) = \sum_{j=1}^{n}\frac{\partial^2\psi_i(t)y}{\partial y_j^2} = \Delta_y\psi_i(t)y \qquad y = \phi(t)x.$$

Proof. Clearly $div_y(\vec{J}) = -k\Delta T(y,t)$ for $y \in \Omega(t)$ and we get (6.1). Now for (6.2), we have from (5.5),

$$\nabla_y T(y,t) = \nabla_x\overline{T}(x,t).D\psi(t)y.$$

Hence,

$$div_y(-k\nabla_y T(y,t)) = -k\ \ div_y(\nabla_x\overline{T}(x,t)D\psi(t)y) =$$

$$-k\ div_y\left(\sum_{i=1}^{n}\frac{\partial\ \overline{T}(x,t)}{\partial x_i}\ .\ \frac{\partial\psi_i(t)y}{\partial y_1},...,\sum_{i=1}^{n}\frac{\partial\ \overline{T}(x,t)}{\partial x_i}\ .\ \frac{\partial\psi_i(t)y}{\partial y_n}\right).$$

Now observe that

$$\frac{\partial}{\partial y_j}\left(\sum_{i=1}^{n}\frac{\partial\ \overline{T}(x,t)}{\partial x_i}.\frac{\partial\ \psi_i(t)y}{\partial y_j}\right)\sum_{i=1}^{n}\left(\frac{\partial}{\partial y_j}\left(\frac{\partial\ \overline{T}(x,t)}{\partial x_i}\right).\frac{\partial\ \psi_i(t)y}{\partial y_j} + \frac{\partial\ \overline{T}(x,t)}{\partial x_i}.\frac{\partial^2\ \psi_i(t)y}{\partial y_j^2}\right) \tag{6.3}$$

and by (5.5), we get

$$\frac{\partial}{\partial y_j}\left(\frac{\partial\ \overline{T}(x,t)}{\partial x_i}\right) = \sum_{k=1}^{n}\frac{\partial^2\ \overline{T}(x,t)}{\partial x_k\ \partial x_i}\frac{\partial\ \psi_k(t)y}{\partial y_j}$$

and we get in (6.3)

$$\sum_{i=1}^{n}\left(\left(\ \sum_{k=1}^{n}\frac{\partial^2\ \overline{T}(x,t)}{\partial x_k\partial x_i}\ .\ \frac{\partial\psi_k(t)y}{\partial y_j}\right)\frac{\partial\psi_i(t)y}{\partial y_j} + \frac{\partial\ \overline{T}(x,t)}{\partial x_i}\ .\ \frac{\partial^2\psi_i(t)y}{\partial y_j^2}\right).$$

Therefore

$$-div_y(-k\nabla_y T(y,t)) = -k\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\ \sum_{k=1}^{n}\frac{\partial^2\ \overline{T}(x,t)}{\partial x_k\partial x_i}\ .\ \frac{\partial\psi_k(t)y}{\partial y_j}\right)\ .\ \frac{\partial\psi_i(t)y}{\partial y_j}\ -$$

$$k\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial\ \overline{T}(x,t)}{\partial x_i}\frac{\partial^2\psi_i(t)y}{\partial y_j^2}$$

which leads to

$$-k\ div_y\left(\sum_{i=1}^{n}\frac{\partial\ \overline{T}(x,t)}{\partial x_i}\cdot\frac{\partial\psi_i(t)y}{\partial y_1},...,\sum_{i=1}^{n}\frac{\partial\ \overline{T}(x,t)}{\partial x_i}\cdot\frac{\partial\psi_i(t)y}{\partial y_n}\right)=$$

$$-k\sum_{k,i=1}^{n}\frac{\partial^2\ \overline{T}(x,t)}{\partial x_k\partial x_i}\cdot\left(\sum_{j=1}^{n}\frac{\partial\psi_k(t)y}{\partial y_j}\cdot\frac{\partial\psi_i(t)y}{\partial y_j}\right)\ -\ k\sum_{i=1}^{n}\frac{\partial\overline{T}(x,t)}{\partial x_i}\cdot\left(\sum_{j=1}^{n}\frac{\partial^2\psi_i(t)y}{\partial y_j^2}\right).$$

and we get the result. ■

Concerning the main part in (6.2) we have the following

Proposition 6.2 *With the notations above, the term*

$$\sum_{k,i=1}^{n}a_{k,i}(x,t)\frac{\partial^2\ \overline{T}}{\partial x_k\partial x_i}(x,t)$$

can be written in divergence form.

Proof. Just note that

$$\sum_{k,i=1}^{n}a_{k,i}(x,t)\frac{\partial^2\ \overline{T}(x,t)}{\partial x_k\partial x_i}=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n}\frac{\partial\overline{T}(x,t)}{\partial x_i}.a_{k,i}(x,t)\right)-\sum_{i=1}^{n}\frac{\partial\overline{T}(x,t)}{\partial x_i}\ .c_i(x,t)$$

with

$$c_i(x,t)=\sum_{k=1}^{n}\frac{a_{k,i}(x,t)}{\partial x_i}$$

■

Remark 6.3 *Note that now (6.2) can be written as*

$$\begin{cases}\frac{\partial\overline{T}(x,t)}{\partial t}+\overline{T}(x,t)\overline{div(\vec{V})}(x,t)-k\left(div(B(x,t))-\sum_{i=1}^{n}\frac{\partial\overline{T}(x,t)}{\partial x_i}d_i(x,t)\right)=\overline{f}(x,t)\\ \\ \overline{T}(x,t)=0\ x\in\partial\Omega_0\quad\forall t\qquad\overline{T}(x,0)=T_0(x)\qquad x\in\Omega_0\end{cases}$$

with

$$\vec{B}=(B_i)_{i=1,..,n}=A(x,t)\nabla_x\overline{T}(x,t)$$
$$A(x,t)=(a_{k,i}(x,t)),\quad d_i(x,t)=s_i(x,t)-c_i(x,t).$$

7 Parabolic PDEs in moving domains

Now we consider general parabolic equations in moving domains. That means that the equations are not necessarily balance equations. Hence, we consider

$$\begin{cases} \frac{\partial}{\partial t}T(y,t) - k\Delta_y T(y,t) + \sum_{i=1}^{n} \frac{\partial T}{\partial y_i}(y,t)\ .\ g_i(y,t) + c(y,t)T(y,t) = f(t,y) \quad y \in \Omega(t) \\ T(y,t) = 0 \quad y \in \partial\Omega(t) \quad \forall t \qquad T(y,0) = T_0(y) \quad y \in \Omega_0 \end{cases} \tag{7.1}$$

with $k > 0$ and given smooth $c(y,t)$ and $\vec{g}(y,t) = (g_i(y,t), \ldots, g_n(y,t))$. Note that this equation contains (6.1) as a particular case.

Then we have the following result whose proof follows from the computation in the sections above.

Proposition 7.1 *With the notations above (7.1) is equivalent to*

$$\begin{cases} \frac{\partial \overline{T}}{\partial t}(x,t) - k div(B(x,t)) + \nabla_x \overline{T}(x,t).\big(\vec{h}(x,t) - \vec{d}(x,t)\big) + \overline{c}(x,t)\overline{T}(x,t) = \overline{f}(x,t) \quad x \in \Omega_0 \\ \overline{T}(x,t) = 0 \quad x \in \partial\Omega_0 \quad \forall t \qquad \overline{T}(x,0) = T_0(x) \quad x \in \Omega_0 \end{cases} \tag{7.2}$$

with

$$B(x,t) = A(x,t)\nabla_x \overline{T}(x,t), \quad A(x,t) = (a_{k,i}(x,t)), \quad a_{k,i}(x,t) = \sum_{j=1}^{n} \frac{\partial \psi_k(t)y}{\partial y_j}\ .\ \frac{\partial \psi_i(t)y}{\partial y_j},$$

$$d_i(x,t) = s_i(x,t) - c_i(x,t), \quad s_i(x,t) = \Delta_y \psi_i(t)y, \quad c_i(x,t) = \sum_{k=1}^{n} \frac{a_{k,i}(x,t)}{\partial x_i}$$

$$\vec{h}(x,t) = (\overline{\vec{g}}(x,t)\ .\ \nabla_y \psi_1(t)y, \ldots, \overline{\vec{g}}(x,t)\ .\ \nabla_y \psi_n(t)y), \quad y = \phi(t)x.$$

Now we are in a position to proof that (7.1) is well posed.

Proposition 7.2 *Under the assumptions above, if the initial data satisfies*

$$T_0 \in L^2(\Omega_0)$$

then (7.2) and (7.1) have a unique solution.

Proof. Observe that in (7.2)

$$A(x,t) = D\psi(t)y.(D\psi(t)y)^t \qquad y = \phi(t)x.$$

Then we show below that this is a positive definite matrix. In fact for $\xi \in \mathbb{R}^n$, $\xi \neq 0$, we have

$$< A(x,t)\xi, \xi >=< (D\psi(t)y)^t\xi, (D\psi(t)y)^t\xi >= \left\|(D\psi(t)y)^t\xi\right\|^2 > 0.$$

since $(D\psi(t)y)^t$ is non singular. Also, from (2.3), the eigenvalues of $D\phi(t)$ are bounded and bounded away from 0 for all $t \in [0, T]$ and so are the eigenvalues of $D\psi(t)$. Therefore there exist $\alpha = \alpha(T) > 0$ such that $\|(D\psi(t)y)^t \xi\|^2 \geq \alpha \|\xi\|^2$.

Using this, the smoothness of the coefficients and the results in [1, 2], we get that (7.2) has a unique smooth solution and so does (7.1). ■

8 Maximum principle

In this section we show that the parabolic equations in moving domains possess the maximum principle. We will show this on the particular example of the heat equation

$$\left\{ \begin{array}{ll} \frac{\partial T}{\partial t}(y,t) - \Delta T(y,t) + a(y,t)\, T(y,t) = 0 & y \in \Omega(t) \\ \\ T(y,t) = 0 \quad y \in \partial\Omega(t) \quad \forall t & T(y,0) = T_0(y) \qquad y \in \Omega_0 \end{array} \right. \tag{8.1}$$

witha a sufficiently smooth coefficient $a(y,t)$. Then we have

Proposition 8.1 *With the assumption above, if*

$$T_0 \in L^2(\Omega_0), \quad T_0(x) \geq 0 \quad x \in \Omega_0$$

and

$$\alpha(t) \leq a(y,t) \quad \forall y \in \Omega(t) \quad \forall t$$

for some smooth $\alpha(t)$. Then

$$T(y,t) \geq 0, \quad y \in \Omega(t), \quad t \geq 0.$$

Proof. We multiply (8.1) by the negative part of T, $T^-(y,t)$, and integrate in $\Omega(t)$, to get

$$\int_{\Omega(t)} \frac{\partial T}{\partial t}(y,t)\, .\, T^-(y,t)\, dy - \int_{\Omega(t)} \Delta T(y,t)\, .\, T^-(y,t)\, dy + \int_{\Omega(t)} a(y,t).\, T(y,t).\, T^-(y,t)\, dy = 0.$$

Using (3.3) for $(T^-)^2$ and the fact that $T^-(y,t) = 0$ in $\partial\Omega(t)$, because $T(y,t) = 0$ in $\partial\Omega(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} (T^-)^2(y,t)\, dy + \int_{\Omega(t)} \left|\nabla T^-(y,t)\right|^2\, dy + \int_{\Omega(t)} a(y,t)(T^-)^2(y,t)\, dy = 0.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\left\|T^-(.,t)\right\|^2_{L^2(\Omega(t))} + \alpha(t)\left\|T^-(.,t)\right\|^2_{L^2(\Omega(t))} \leq 0$$

and taking $\bar{F}(t) = \|T^-(.,t)\|^2_{L^2(\Omega(t))}$, we have

$$\frac{d}{dt}\bar{F}(t) + 2\alpha(t)\bar{F}(t) \leq 0$$

and Gronwall's lemma leads to $\bar{F}(t) \leq \left\|T_0^-\right\|_{L^2(\Omega_0)} e^{-2\int_0^t \alpha(s)\, ds} = 0$, since $T_0^- = 0$ in Ω_0. Therefore $T^-(y,t) = 0$ for $y \in \Omega(t)$ and $t \geq 0$ as claimed. ■

9 Energy estimates

In this section we derive suitable energy estimates for the heat equation in a moving domain

$$\begin{cases} \frac{\partial T}{\partial t}(y,t) - \Delta T(y,t) + a(y,t)\ T(y,t) = 0 \quad y \in \Omega(t) \\ T(y,t) = 0 \quad y \in \partial\Omega(t) \quad \forall t \qquad T(y,0) = T_0(y) \qquad y \in \Omega_0 \end{cases} \tag{9.1}$$

with a smooth enough $a(y,t)$. First, we have for nonnegative solutions

Proposition 9.1 *Assume*

$$T_0 \in L^2(\Omega_0) \quad T_0(x) \geq 0, \quad x \in \Omega_0$$

and

$$\alpha(t) \leq a(y,t) \quad \forall y \in \Omega(t) \quad \forall t.$$

for some smooth $\alpha(t)$ *such that*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \alpha(s)\,ds > \alpha_0 > 0.$$

Then

$$\int_{\Omega(t)} T(y,t)\,dy \leq e^{-\int_0^t \alpha(t)\,ds} \int_{\Omega_0} T_0(x)\,dx \xrightarrow[t\to\infty]{} 0.$$

Proof. From (3.3)

$$\frac{d}{dt}\int_{\Omega(t)} T(y,t)\,dy = \int_{\Omega(t)} \frac{\partial T}{\partial t}(y,t)\,dy + \int_{\partial\Omega(t)} T(y,t)\vec{V}(y)\,d\vec{s}$$

and since T vanishes on the boundary, we have

$$\frac{d}{dt}\int_{\Omega(t)} T(y,t)\,dy = \int_{\Omega(t)} \frac{\partial T}{\partial t}(y,t)\,dy.$$

Using this, we integrate in (9.1) in $\Omega(t)$, to get

$$\int_{\Omega(t)} \frac{\partial T}{\partial t}(y,t)\,dy - \int_{\Omega(t)} \Delta T(y,t)\,dy + \int_{\Omega(t)} a(y,t)\ T(y,t)\,dy = 0.$$

Now Green's formula leads to

$$\frac{d}{dt}\int_{\Omega(t)} T(y,t)\,dy - \int_{\partial\Omega(t)} \frac{\partial T}{\partial \vec{n}}(y,t)\,d\vec{s} + \int_{\Omega(t)} a(y,t)\ T(y,t)\,dy = 0.$$

By the maximum prinicple we know that $T(y,t) \geq 0$ for $y \in \Omega(t)$ and $t \geq 0$, and then for $y \in \partial\Omega(t)$ we have $\frac{\partial T}{\partial \vec{n}}(y,t) \leq 0$ and then

$$\frac{d}{dt}\int_{\Omega(t)} T(y,t)\,dy + \int_{\Omega(t)} a(y,t)\ T(y,t)\,dy \leq 0.$$

Hence, denoting $\bar{Y}(t) = \int_{\Omega(t)} T(y,t)\,dy$ we have

$$\frac{d\bar{Y}}{dt}(t) + \alpha(t)\bar{Y}(t) \le 0$$

and Gronwall's lemma gives

$$\bar{Y}(t) = \int_{\Omega(t)} T(y,t)\,dy \le e^{-\int_0^t \alpha(t)\,ds} \int_{\Omega_0} T_0(x)\,dx \xrightarrow[t\to\infty]{} 0.$$

since, by assumption

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \alpha(s)\,ds > \alpha_0 > 0$$

and then

$$e^{-\int_0^t \alpha(s)\,ds} = e^{-t\left(\frac{1}{t}\int_0^t \alpha(s)\,ds\right)} \le e^{-\alpha_0 t} \xrightarrow[t\to\infty]{} 0.$$

for $t >> 1$. ■

Now without assuming sign on the solutions, we have

Proposition 9.2 *With the notations above, assume*

$$T_0 \in L^2(\Omega_0)$$

and the function

$$\gamma(t) = \alpha(t) - C_0(\Omega(t)),$$

is such that for some $\alpha_1 > 0$*,*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \gamma(s)\,ds > \alpha_1 > 0,$$

where $C_0(\Omega(t))$ *is the Poncairè constant in* $\Omega(t)$*.*

Then

$$0 \le \int_{\Omega(t)} T^2(y,t)\,dy \le e^{-2\int_0^t \gamma(s)\,ds} \int_{\Omega(t)} T_0^2(x)\,dx \xrightarrow[t\to\infty]{} 0.$$

Proof. Multiply (9.1) by $T(y,t)$ and integrate in $\Omega(t)$, to get

$$\int_{\Omega(t)} \frac{\partial T}{\partial t}(y,t)T(y,t)\,dy - \int_{\Omega(t)} \Delta T(y,t)T(y,t)\,dy + \int_{\Omega(t)} a(y,t)\ T^2(y,t)\,dy = 0.$$

Using (3.3), the boundary conditions and the Green's formula we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega(t)} T^2(y,t)\,dy + \int_{\Omega(t)} |\nabla T(y,t)|^2\,dy + \int_{\Omega(t)} a(y,t)T^2(y,t)\,dy = 0.$$

Now the Poincarè inequality in $\Omega(t)$ gives for any smooth function vanishing on $\partial\Omega(t)$,

$$\|\nabla u\|^2_{L^2(\Omega(t))} \geq C_0(\Omega(t))\, \|u\|^2_{L^2(\Omega(t))}\,.$$

This and the assumption on $a(y,t)$ leads to

$$\frac{1}{2}\frac{d}{dt}\,\|T(.,t)\|^2_{L^2(\Omega(t))} + \gamma(t)\,\|T(.,t)\|^2_{L^2(\Omega(t))} \leq 0. \tag{9.2}$$

Thus, denoting $\bar{Z}(t) = \|T(.,t)\|^2_{L^2(\Omega(t))}$, (9.2) reads

$$\frac{d}{dt}\bar{Z}(t) + 2\gamma(t)\bar{Z}(t) \leq 0$$

and Gronwall's lemma yields

$$\bar{Z}(t) \leq \|T_0\|^2_{L^2(\Omega_0)}\, e^{-2\int_0^t \gamma(s)\,ds} \xrightarrow[t\to\infty]{} 0.$$

since by assumption

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \gamma(s)\,ds > \alpha_1 > 0$$

and then

$$e^{-\int_0^t \gamma(s)\,ds} = e^{-t\left(\dfrac{1}{t}\displaystyle\int_0^t \gamma(s)\,ds\right)} \leq e^{-\alpha_1 t} \xrightarrow[t\to\infty]{} 0$$

for $t >> 1$. ■

References

[1] Amann, H. *Linear and Quasilinear Parabolic Problems*, **89**, Birkhauser Verlag, Berlin (1995).

[2] Amann, H. *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems Function spaces, differential operators and nonlinear analysis*, (Friedrichroda, 1992), Teubner, 1993, 133, 9-126

[3] Duvaut, G. *Mécanique des milieux continus*, Masson, 1990

[4] Hartman, P. *Ordinary differential equations*, John Wiley and sons, 1964.

[5] Chorin, A. J., Marsden J.E., *A mathematical introduction to fluid mechanics*, Springer, 1979.

[6] Ladyzhenskaya, O. A., *The mathematical theory of viscous incompressible flow*, Gordon and Breach, 1969.